Advanced Model Theory

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1: Preliminaries

Two running examples:

1. Theory VSP_K^∞ of infinite vector spaces over a fixed field K: models are determined by dimension. In particular, exactly one model (up to $\cong$) in each uncountable cardinality κ (i.e., κ-categorical).
2. Theory ACF_p of algebraically closed fields of characteristic p: models are determined by transcendence degree. In particular, exactly one model in each uncountable cardinality.

Similarity of arguments – find general argument? General notion of "dimension"? Goal: develop such a general theory and prove:

Morley's theorem Let $\kappa > \aleph_0$. A theory in an at most countable language is κ-categorical if and only if it is $\aleph_1$-categorical.

Basic notations, elementary maps, elementary diagrams

2: Types

Definition and characterizations of types, realizability
type space, Stone topology
homeomorphisms from elementary maps.

3: Saturation

elementary chains and Tarski's lemma
existence of κ^+-saturated elementary extensions
back and forth: isomorphism from saturation

κ-categorical iff all size κ models are saturated: interesting mistake for "if" and uncountable κ

4: Crash course: quantifier elimination

sufficient for primitive existential formulas
quantifier elimination criterion
Examples VSP_K^∞ and ACF_p

5: Indiscernibles

Ehrenfeucht-Mostiowski types
Ramsey's theorem
Indiscernible existence lemma
models generated by well-ordered indiscernibles realize few types
Few Types theorem

6: ω-stability

κ-stable theories
Example: ACF_p
$\aleph_0 < \kappa$ -categorical implies ω-stable
ω-stable implies totally transcendental implies κ-stable for all $\kappa > |L|$
κ-stable gives λ^+-saturated model of size κ (all $\lambda < \kappa$)
correction of interesting mistake above

7: Prime extensions

prime extensions and constructible extensions of parameter sets
t.t. theories: isolated types are dense
t.t. theories: existence of construcible extensions
t.t. theories: prime implies atomic
Lachlan's lemma
Morley downwards

8: Crash course: ω-homogeneity

existence of small ω-homogeneous extensions
back and forth: isomorphism from homogeneity
algebraic characterization of type equality
Beth's definability theorem

9: Vaughtian pairs

no VP implies elimination of $\exists^\infty$
Vaught's two cardinal theorem
$\aleph_0 < \kappa$-categorical implies no VP

10: Matroids

Definition of independence and bases in matroids
Examples: vector spaces, algebraically closed fields
Bases exist, their cardinality is unique: dimension
dimension formula for restricted and relativized matroids

11: Algebraicity

algebraic formulas and types
transitivity of algebraic closure
existence of non-algebraic type extensions

12: Strong minimality

minimal and strongly minimal formulas
coincide under ω-saturation or $\exists^\infty$-elimination
existence of minimal formulas for t.t. theories
Examples: ACF_p and VSP_K^∞ are strongly minimal theories
strongly minimal types have unique non-algebraic extensions
indiscernibility from independence
algebraic closure gives matroid on strongly minimal sets
Example: for ACF_p dimension is transcendence degree
φ-dimensions
extension of elementary maps to algebraic closures
Strongly minimal theories:

- models determined by dimension
- model dimensions form an endsegment of cardinals
- ω-saturation means infinite dimension
- all models are ω-homogeneous
- κ-categorical for all $\kappa > \aleph_0$

Baldwin-Lachlan theorem
Morley's theorem