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Fourier Series on Certain Solenoids

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1. Introduction

(1.1) *Notation.* We use the symbols $\mathbb{N}$, $\mathbb{Z}^+$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ to designate the positive integers, nonnegative integers, integers, rational numbers, real numbers, and complex numbers, respectively. We denote the circle group by $\mathbb{T}$ and will customarily parametrize this group as $\{\exp(2\pi it) : -\frac{1}{2} \leq t < \frac{1}{2}\}$. For real numbers a and b such that $a \leq b$, let $[a, b]$ be the closed interval $\{t : t \in \mathbb{R}, a \leq t \leq b\}$, and let $]a, b[$ be the open interval $\{t : t \in \mathbb{R}, a < t < b\}$. The intervals $[a, b[$ and $]a, b]$ are defined similarly. Given a real number t, the symbol $[t]$ denotes the largest integer not exceeding t.

Throughout the paper, G will denote a compact Abelian group, and the word "character" will mean "continuous character". We denote the character group of G by X. The symbols $\mathfrak{C}(G)$, $\mathfrak{L}_p(G)$, and $\mathbf{M}(G)$ denote the set of all continuous, complex–valued functions on G, the set of Haar measurable functions on G with absolutely integrable p^{th} powers ($1 \leq p < \infty$), and the set of all complex–valued countably additive Borel measures on G, respectively.

A *trigonometric polynomial* Φ on G is a function of the form

$$\Phi = \sum a_\chi \chi,$$

where a is a function on X that vanishes off of a finite subset of X. The set of $\chi \in \mathsf{X}$ for which $a_\chi \neq 0$ is called *the set of frequencies of* Φ.

Given a set X and a subset A of X, we write the complement of A in X as $X \setminus A$ or sometimes as A', where there is no doubt as to what the set X is. The indicator function of A, written as 1_A, is the function on X equal to 1 on A and equal to 0 on $X \setminus A$. (In every case, it will be clear from the context what the set X is.)

(1.2) *The subgroups of* $\mathbb{Q}$. Recall that G is a *solenoid* if G contains a continuous homomorphic image of the additive group $\mathbb{R}$ that is dense in G, and also that G is a solenoid if and only if (a) X is torsion–free and (b) $\mathrm{card}(\mathsf{X}) \leq 2^{\aleph_0}$. (All of this is described in [10], Theorem (25.18)). The solenoids we study in this paper are of a very special sort.

Let A be any noncyclic subgroup of the additive group $\mathbb{Q}$. Plainly the character group of A is a particularly simple solenoid. The group A can be described in various ways. For our purposes, we need the following description of A. Throughout the paper, let $\mathbf{a} = (a_0, a_1, \ldots, a_n, \ldots)$ be an arbitrary but fixed sequence of integers all greater than 1. Given such a sequence $\mathbf{a}$, we define:

$$A_0 = 1; A_k = a_0 a_1 \ldots a_{k-1} \qquad \text{for } k \in \mathbb{N}. \tag{1}$$

Let $\mathbb{Q}_{\mathbf{a}}$ be the set of all rational numbers l/A_k, as l runs through $\mathbb{Z}$ and k runs through $\mathbb{Z}^+$. Plainly $\mathbb{Q}_{\mathbf{a}}$ is a subgroup of the additive group $\mathbb{Q}$; since all of the integers a_j exceed 1, $\mathbb{Q}_{\mathbf{a}}$ is noncyclic.

Beaumont and Zuckerman [3] have shown that every noncyclic subgroup A of $\mathbb{Q}$ is isomorphic to some group $\mathbb{Q}_{\mathbf{a}}$ as described above. To tell when two groups $\mathbb{Q}_{\mathbf{a}}$ and $\mathbb{Q}_{\mathbf{b}}$ are isomorphic, proceed as follows. For all primes p, let $k_{\mathbf{a}}(p)$ be defined by:

$$k_{\mathbf{a}}(p) = \begin{cases} 0 \text{ if } p \text{ divides no } A_k; \\ m \in \mathbb{N} \text{ if } p^m \text{ divides some} A_k \text{ and } p^{m+1} \text{ divides no } A_k; \\ \infty \text{ otherwise.} \end{cases}$$

Beaumont and Zuckerman (loc.cit.) prove that $\mathbb{Q}_{\mathbf{a}}$ and $\mathbb{Q}_{\mathbf{b}}$ are isomorphic if and only if: (a) $k_{\mathbf{a}}(p) = k_{\mathbf{b}}(p)$ for all but finitely many values of p; and (b) $k_{\mathbf{a}}(p) \neq k_{\mathbf{b}}(p)$ implies that both are finite. Plainly a continuum of different sequences $\mathbf{a}$ will yield the same subgroup of $\mathbb{Q}$. Note too that $\mathbb{Q}_{\mathbf{a}}$ is isomorphic with $\mathbb{Q}$ only when $\mathbb{Q}_{\mathbf{a}}$ is actually $\mathbb{Q}$; and this occurs if and only if $k_{\mathbf{a}}(p) = \infty$ for all primes p.

(1.3) *The character group of* $\mathbb{Q}_{\mathbf{a}}$. We will study Fourier series (suitably defined) on the character group of $\mathbb{Q}_{\mathbf{a}}$, $\mathbb{Q}_{\mathbf{a}}$ being given the discrete topology. Thus its character group is a compact Abelian group. We will use a particular realization of this group. First we form the group $\Delta_{\mathbf{a}}$ of **a**–*adic integers*. This group which is described in [10], (10.1), consists of all sequences $\mathbf{x} = (x_0, x_1, \ldots, x_n, \ldots)$ with $x_j \in \{0, 1, 2, \ldots, a_j - 1\}$. We give $\Delta_{\mathbf{a}}$ the Cartesian product topology, each "coordinate set" $\{0, 1, 2, \ldots, a_j - 1\}$ being discrete. We define the sum $\mathbf{z} = \mathbf{x} + \mathbf{y}$ of elements of $\Delta_{\mathbf{a}}$ by induction. Write $x_0 + y_0 = z_0 + a_0 q_0$, where $z_0 \in \{0, 1, \ldots, a_0 - 1\}$ and $q_0 \in \{0, 1\}$. If $z_0, z_1, \ldots, z_{n-1}$ and $q_0, q_1, \ldots, q_{n-1}$ have been defined, write $x_n + y_n + q_{n-1} = z_n + a_n q_n$, where $z_n \in \{0, 1, \ldots, a_n - 1\}$ and $q_n \in \{0, 1\}$. It is easy to verify that $\Delta_{\mathbf{a}}$ is a compact 0–dimensional Abelian group, with normalized Haar measure, say λ_0, the product measure of the measures assigning the measure $1/a_n$ to each point of $\{0, 1, \ldots, a_{n-1}\}$. Let $\mathbf{u}$ be the element $(1, 0, 0, \ldots, 0, \ldots)$ of $\Delta_{\mathbf{a}}$.

Next consider the product group $\mathbb{R} \times \Delta_{\mathbf{a}}$ and in it the infinite cyclic discrete subgroup $H = \{(n - n\mathbf{u}) : n \in \mathbb{Z}\}$. The quotient group $(R \times \Delta_{\mathbf{a}})/H$ is denoted by the symbol $\Sigma_{\mathbf{a}}$, and is called the **a**–*adic solenoid*. The group $\Sigma_{\mathbf{a}}$ can be conveniently realized as the set $[-\frac{1}{2}, \frac{1}{2}[\times \Delta_{\mathbf{a}}$, which is of course a subset of $\mathbb{R} \times \Delta_{\mathbf{a}}$. We will denote this set by $\Sigma_{\mathbf{a}}$. Every coset of H in $\mathbb{R} \times \Delta_{\mathbf{a}}$ contains exactly one element of $\Sigma_{\mathbf{a}}$. The sum $(s, \mathbf{x}) + (t, \mathbf{y})$ of two elements of $\Sigma_{\mathbf{a}}$ is the element $(s + t - [s + t + \frac{1}{2}], [s + t + \frac{1}{2}]\mathbf{u} + \mathbf{x} + \mathbf{y})$. The topology of $\Sigma_{\mathbf{a}}$ is defined by a complete family of neighborhoods $\{U_k : k \in \mathbb{N}\}$ of the neutral element $(0, \mathbf{0})$:

$$U_k = \{(s, \mathbf{x}) \in \Sigma_{\mathbf{a}} : -\frac{1}{2k} < s < \frac{1}{2k} \quad \text{and} \quad x_0 = x_1 = \cdots = x_{k-1} = 0\}.$$

Note that neighborhoods of the points $(-\frac{1}{2}, \mathbf{x})$ have the form

$$\{(-\frac{1}{2} + t, \mathbf{y}) : 0 \leq t < \frac{1}{2k} \text{ and } y_0 = x_0, y_1 = x_1, \ldots, y_{k-1} = x_{k-1}\} \cup$$
$$\{(\frac{1}{2} - t, -\mathbf{u} + \mathbf{y}) : 0 \leq t < \frac{1}{2k} \text{ and } y_0 = x_0, y_1 = x_1, \ldots, y_{k-1} = x_{k-1}\}.$$

The foregoing description of $\Sigma_{\mathbf{a}}$ differs only in minor details from that given in [10], (10.12)–(10.15).

Normalized Haar measure μ on $\Sigma_{\mathbf{a}}$ is the product of Lebesgue measure λ on $[-\frac{1}{2}, \frac{1}{2}[$ and the Haar measure λ_0 on $\Delta_{\mathbf{a}}$ referred to above.

(1.4) *The character groups of* $\Delta_{\mathbf{a}}$ *and* $\Sigma_{\mathbf{a}}$. For $\mathbf{x} \in \Delta_{\mathbf{a}}$ and $\alpha = \frac{1}{A_j} \in \mathbb{Q}_{\mathbf{a}}$, we write

$$\chi_\alpha(\mathbf{x}) = \exp\left[2\pi i \frac{l}{A_j}(x_0 + A_1x_1 + A_2x_2 + \cdots + A_{j-1}x_{j-1})\right]. \tag{1}$$

It is known [10], (25.2), and easy to verify that each χ_a is a character of $\Delta_{\mathbf{a}}$, that $\chi_\alpha\chi_\beta = \chi_{\alpha+\beta}$, that all characters of $\Delta_{\mathbf{a}}$ have the form (1), and that $\chi_\alpha = \chi_\beta$ if and only if $\alpha - \beta$ is an integer. That is, the character group of $\Delta_{\mathbf{a}}$ can be identified with the discrete torsion group $\mathbb{Q}_{\mathbf{a}}/\mathbb{Z}$.

For $(t, \mathbf{x}) \in \Sigma_{\mathbf{a}}$ and $\alpha = \frac{l}{A_j} \in \mathbb{Q}_{\mathbf{a}}$, we write

$$\begin{aligned} \chi_\alpha(t, \mathbf{x}) &= \exp\left[2\pi i \frac{l}{A_j}(t + x_0 + A_1x_1 + A_2x_2 + \cdots + A_{j-1}x_{j-1})\right] \\ &= \exp\left[2\pi i \frac{l}{A_j}\left(t + \sum_{k=0}^{\infty} A_k x_k\right)\right], \end{aligned} \tag{2}$$

where we agree that integer terms in the series $\frac{l}{A_j}\sum_{k=0}^{\infty} A_k x_k$ are to be omitted. As shown in [10], (25.3), these functions are exactly the characters of the group $\Sigma_{\mathbf{a}}$. It is easy to see that $\chi_\alpha\chi_\beta = \chi_{\alpha+\beta}$ and that $\chi_\alpha = \chi_\beta$ if and only if $\alpha = \beta$. That is, the character group of $\Sigma_{\mathbf{a}}$ can be identified with the group $\mathbb{Q}_{\mathbf{a}}$. We use the same notation in (1) and (2): the context will always make clear which group we are using.

(1.5) *Fourier transforms on* $\Sigma_{\mathbf{a}}$. Once again we consider an arbitrary G. For $f \in \mathfrak{L}_1(G)$ and $\chi \in \mathsf{X}$, write

$$\widehat{f}(\chi) = \int_G f(t)\overline{\chi(t)}d\nu(t), \tag{1}$$

where ν is normalized Haar measure on G. The function $\widehat{f}$ on X defined by (1) is of course the *Fourier transform* of f. The theory of Fourier transforms on compact Abelian groups and locally compact Abelian groups has been studied intensively for nearly half a century. The reader may consult [20, 10, 11]. We will be largely concerned throughout this paper with the case $G = \Sigma_{\mathbf{a}}$ and $\mathsf{X} = \mathbb{Q}_{\mathbf{a}}$, connected by the definition (1.4.2).

(1.6) *Fourier series on compact Abelian groups.* The classical theory of Fourier series on the group $\mathbb{T}$ deals with the behavior of the sequences of functions $S_nf(t)$, defined for $f \in \mathfrak{L}_1(\mathbb{T})$ by

$$S_nf(t) = \sum_{k=-n}^{n} \widehat{f}(k)\exp(2\pi ikt), \tag{1}$$

for all $n \in \mathbb{Z}^+$ and $t \in [-\frac{1}{2}, \frac{1}{2}[$. Classical Fourier series have occupied a central place in analysis for well over two centuries. The contemporary theory is set forth in the great treatise [24]. In many cases of interest where the series (1) converges, it does so only conditionally, and so the order of summation is vitally important. We may look for analogues of the series (1) for infinite compact metric Abelian groups G different from $\mathbb{T}$. To do this, we must find a reasonable way of arranging the character group X into increasing blocks, say Γ_n in such a way that:

$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n \subset \ldots; \tag{2}$$

$$\text{each } \Gamma_n \text{ is finite;} \tag{3}$$

$$\bigcup_{n=0}^{\infty} \Gamma_n = \mathsf{X}. \tag{4}$$

The difficulties that attend such a program are apparent even in the apparently simple case of multiple Fourier series, which is the case $G = \mathbb{T}^l$ with $l \in \{2, 3, 4, \dots\}$. (For an interesting introduction to this theory, see [1].)

Once the sets Γ_n have been chosen, we define *the Fourier series of a function* f in $\mathfrak{L}_1(G)$ as the series whose n^{th} partial sum is the polynomial

$$S_n f(t) = \sum_{\chi \in \Gamma_n} \widehat{f}(\chi)\chi(t) \tag{5}$$

One can then ask a large number of questions, for example: (a) does a given Fourier series converge, and if so, to what function?; (b) is there a summability method for the Fourier series giving convergence almost everywhere on G to the function, in analogy with the theorem of Fejér–Lebesgue?; (c) is there an analogue of the theorem of Riesz [19] giving inequalities $\|S_n f\|_p \le A_p \|f\|_p$ for $f \in \mathfrak{L}_p(G)$ $(1 < p < \infty)$?; (d) is there a uniqueness theorem for convergent *trigonometric* series on G that are not necessarily Fourier series?; and so on. In short, one may attempt to "rewrite Zygmund" with its treasure house of delicate facts for the group G. (We refer of course to [25].)

The success of the above program, or any significant part of it, must depend upon a shrewd choice of the sets Γ_n. A large number of writers have studied such sets Γ_n for one or another class of groups and have answered various of the questions (a)–(d). The earliest success known to us in this program was achieved by Paley [18], who ordered the Walsh functions (which are essentially the characters of the dyadic group $\{0, 1\}^{\aleph_0}$ in a particularly felicitous way and obtained a number of delicate facts about the partial sums of Walsh–Fourier series. (Walsh, who constructed the Walsh functions from the Rademacher functions in [23], also ordered the Walsh functions, but not in the same way as Paley.) Billard [4] and Sjölin [21] have used Paley's ordering to achieve very precise results on the almost everywhere convergence of Fourier series on the group $\{0, 1\}^{\aleph_0}$.

Vilenkin [22] obtained a whole class of orderings for the characters of an arbitrary metrizable 0–dimensional infinite compact Abelian group. Under his oderings, a number of standard facts about classical Fourier series admit analogues. Vilenkin's construction admits Paley's as a special case. Moore [17] used one of Vilenkin's orderings to obtain convergence theorems for Fourier series on a countably infinite product of cyclic groups of bounded orders. Hunt and Taibleson [13] studied the compact additive group of integers in a totally disconnected locally compact nondiscrete field, imposing an order on its characters under which the Carleson–Hunt theorem holds. Gosselin [7] has obtained the Carleson–Hunt theorem and Riesz's theorem on $\mathfrak{L}_p$ norms of partial sums of Fourier series for a large class of 0–dimensional compact Abelian metric groups. His results contain those of Hunt and Taibleson, Sjölin, Billard, and Moore.

2. Fourier Series on $\Sigma_{\mathbf{a}}$

(2.1) *Definitions.* We return to our principal object of interest, the group $\Sigma_{\mathbf{a}}$ defined in (1.3). For this group and its character group $\mathbb{Q}_{\mathbf{a}}$, we make a selection of sets Γ_n as follows. Let $(c_j)_{j=0}^{\infty}$ be a sequence of positive integers for which the properties

$$\lim_{j \to \infty} \frac{c_j}{A_j} = \infty \tag{1}$$

and

$$\frac{c_{j+1}}{A_{j+1}} > \frac{c_j}{A_j} \qquad \text{for } j = 0, 1, 2, \dots \tag{2}$$

hold.

The j^{th} *complete block* in C_j in $\mathbb{Q}_{\mathbf{a}}$ is defined by

$$C_j = \left\{ \frac{l}{A_j} : l \in \mathbb{Z}, |l| \le c_j \right\}, \qquad \text{for } j = 0, 1, 2, \ldots \tag{3}$$

The j^{th} *first entrance block* B_j is defined by

$$B_0 = C_0, B_j = C_j \setminus C_{j-1} \qquad \text{for } j = 1, 2, 3, \ldots \tag{4}$$

Plainly (1.6.2)–(1.6.4) hold for the blocks C_j. These blocks have been used by Hewitt and Katznelson [9] to study uniform distributions on $\Sigma_{\mathbf{a}}$ for the case in which $\mathbb{Q}_{\mathbf{a}}$ is a *subring* of $\mathbb{Q}$ (i.e., all nonzero $k_{\mathbf{a}}(p)$ are ∞).

For distinct positive α and β in $\mathbb{Q}_{\mathbf{a}}$, we write $\alpha \prec \beta$ if:

$$\begin{aligned} &\alpha \text{ and } \beta \text{ are in the same first entrance block } B_m \text{ and} \\ &\alpha < \beta \text{ in the usual ordering of } \mathbb{Q} \end{aligned} \tag{5}$$

or

$$\alpha \in B_l \text{ and } \beta \in B_m \text{ with } l < m. \tag{6}$$

That is, the ordering $\prec$ is carried out as follows. We first list all of the positive integers in $B_0 = C_0$ in their natural order; then all positive nonintegers l/A_1 not exceeding c_0 in their natural order; then all l/A_1 such that $c_0 < l/A_1 \le c_1/A_1$ in their natural order; and so on. We plainly get a one–to–one map $n \mapsto \alpha_n$ of $\mathbb{N}$ onto the positive numbers in $\mathbb{Q}_{\mathbf{a}}$ such that $m < n$ if and only if $\alpha_m \prec \alpha_n$. Finally we set $\alpha_0 = 0$ and $\alpha_{-n} = -\alpha_n$.

(2.2) *Definitions.* Let ϕ be a complex–valued function on $\mathbb{Q}_{\mathbf{a}}$. For every nonnegative integer n, let $s_n\phi$ be the polynomial on $\Sigma_{\mathbf{a}}$ defined by

$$s_n\phi(t, \mathbf{x}) = \sum_{l=-n}^{n} \phi(\alpha_l)\chi_{\alpha l}(t, \mathbf{x}). \tag{1}$$

The functions $s_n\phi$ represent a general *trigonometric series* on $\Sigma_{\mathbf{a}}$. If ϕ is the Fourier transform $\widehat{f}$ of a function f in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$, we change the notation of (1) for historical reasons and write

$$s_n f(t, \mathbf{x}) = \sum_{l=-n}^{n} \widehat{f}(\chi_{\alpha_l})\chi_{\alpha l}(t, \mathbf{x}). \tag{2}$$

The function $s_n f$ is called the n^{th} *partial sum of the Fourier series of the function* f. We will devote our attention to a subsequence of the sequence of functions $(s_n f)_{n=0}^{\infty}$. This is defined by

$$S_j f(t, \mathbf{x}) = \sum_{\alpha \in C-j} \widehat{f}(\chi_\alpha)\chi_\alpha(t, \mathbf{x}) \qquad \text{for } j = 0, 1, 2, \ldots \tag{3}$$

That is, we obtain the polynomial $S_j f$ by summing over the j^{th} complete block of characters. It is plain that

$$S_j f = s_{c_j} f. \tag{4}$$

3. More on the Structure of $\Sigma_{\mathbf{a}}$

(3.1) *A construction.* It is essential for our analysis to consider the subgroups Λ_j of $\Sigma_{\mathbf{a}}$ defined by

$$\Lambda_j = \{(0, \mathbf{x}) \in \Sigma_{\mathbf{a}} : x_0 = x_1 = \cdots = x_{j-1} = 0\} \tag{1}$$

for $j = 1, 2, 3, \dots$, and by

$$\Lambda_0 = \{(0, \mathbf{x}) \in \Sigma_{\mathbf{a}} : \mathbf{x} \text{ is arbitrary in } \Delta_{\mathbf{a}}\}. \tag{2}$$

(This notation differs from that used in [10], (10.4)). The sets Λ_j are obviously closed subgroups of $\Sigma_{\mathbf{a}}$. For all j, the quotient group $\Sigma_{\mathbf{a}}/\Lambda_j$ is topologically isomorphic with the circle group $\mathbb{T}$. The natural mapping π_j of $\Sigma_{\mathbf{a}}$ onto $\Sigma_{\mathbf{a}}/\Lambda_j \simeq \mathbb{T}$ can be conveniently realized as

$$\pi_j(t, \mathbf{x}) = \chi_{1/A_j}(t, \mathbf{x}) = \exp\left[2\pi i \frac{1}{A_j}\left(t + \sum_{h=0}^{j-1} A_h x_h\right)\right] \tag{3}$$

for $j = 1, 2, 3, \dots$ and for $j = 0$ as

$$\pi_0(t, \mathbf{x}) = \chi_1(t, \mathbf{x}) = \exp[2\pi i t]. \tag{4}$$

To verify this, note first that the annihilator in $\mathbb{Q}_{\mathbf{a}}$ of the subgroup Λ_j is exactly the cyclic subgroup $\frac{1}{A_j}\mathbb{Z}$ of $\mathbb{Q}_{\mathbf{a}}$. Thus π_j is a continuous homomorphism of $\Sigma_{\mathbf{a}}$ onto $\mathbb{T}$ with kernel Λ_j. Hence it induces a one–to–one map of $\Sigma_{\mathbf{a}}/\Lambda_j$ onto $\mathbb{T}$, which we also denote by the symbol π_j. The definition of the topology of the quotient group $\Sigma_{\mathbf{a}}/\Lambda_j$ (see for example [10], (5.15)) shows that π_j maps $\Sigma_{\mathbf{a}}/\Lambda_j$ continuously onto $\mathbb{T}$. Since $\Sigma_{\mathbf{a}}/\Lambda_j$ and $\mathbb{T}$ are compact, π_j is a topological isomorphism. More specifically, we note as well that:

$$\begin{aligned}&\text{the } j+1\text{–tuples } (t; x_0, x_1, \dots, x_{j-1}) \text{ for } -\frac{1}{2} \le t < \frac{1}{2} \text{ and } x_h \in \{0, \dots, a_h - 1\} \\ &\text{are in one–to–one correspondence with the cosets of } \Lambda_j \text{ in } \Sigma_{\mathbf{a}}.\end{aligned} \tag{5}$$

We shall also have occasion to use normalized Haar measure λ_j on Λ_j, regarded as a measure in $\mathbf{M}(\Sigma_{\mathbf{a}})$. Note that $\|\lambda_j\| = \lambda_j(\Sigma_{\mathbf{a}}) = \lambda_j(\Lambda_j) = 1$.

(3.2) *Remark.* Let p be any polynomial on the group $\mathbb{T}$:

$$p(\exp[2\pi i t]) = \sum_{k=m}^{n} a_k \exp[2\pi i t]. \tag{1}$$

For every nonnegative integer j, the function $p \circ \pi_j$ is plainly a polynomial on $\Sigma_{\mathbf{a}}$:

$$p \circ \pi_j(t, \mathbf{x}) = \sum_{k=m}^{n} a_k \exp\left[2\pi i k \frac{1}{A_j}\left(t + \sum_{h=0}^{\infty} A_h x_h\right)\right] = \sum_{k=m}^{n} a_k \chi_{k/A_j}(t, \mathbf{x}). \tag{2}$$

For our construction of divergent Fourier series on $\Sigma_{\mathbf{a}}$, we need detailed information about measurable subsets of $\Sigma_{\mathbf{a}}$.

(3.3) *Definition.* For a finite subset F of $\mathbb{Z}^+$ and an element $\mathbf{c}$ of the Cartesian product $\mathbf{P}_{j\in F}\{0, 1, \dots, a_j - 1\}$, let $C(F, \mathbf{c})$ be the set of all $\mathbf{x} \in \Delta_{\mathbf{a}}$ such that $x_j = c_j$ for all $j \in F$. Such sets are called *cylinders in* $\Delta_{\mathbf{a}}$. The entire group $\Delta_{\mathbf{a}}$ is a cylinder with F the void set. Let H be any interval in $[-\frac{1}{2}, \frac{1}{2}[$ of the form $[a, b[$ with $-\frac{1}{2} \le a < b < \frac{1}{2}$. A subset of $\Sigma_{\mathbf{a}}$ of the form $[a, b[\times C$, where C is a cylinder in $\Delta_{\mathbf{a}}$, is called an *interval in* $\Sigma_{\mathbf{a}}$.

(3.4) **Theorem.** *Every open subset U of $\Sigma_{\mathbf{a}}$ is the union of a countable family of pairwise disjoint intervals in $\Sigma_{\mathbf{a}}$.**

*Note that intervals in $\Sigma_{\mathbf{a}}$ as we have defined them are *not* open subsets of $\Sigma_{\mathbf{a}}$. Thus the present theorem is not an analogue of the classical structure theorems for open subsets of $\mathbb{R}$ and $\mathbb{T}$. Note also that the decomposition obtained in the present theorem is by no means unique. The open set $]-\frac{1}{4}, \frac{1}{4}[\times \Delta_{\mathbf{a}}$ for example is the union $\bigcup_{k=1}^{\infty}[t_k, t_{k-1}[\times\Delta_{\mathbf{a}}$ for every strictly decreasing sequence of real numbers $(t_k)_{k=0}^{\infty}$ such that $t_0 = \frac{1}{4}$ and $\lim_{k\to\infty} t_k = -\frac{1}{4}$

Proof. Every open subset of $\Sigma_{\mathbf{a}}$ is a union of intervals $H \times C$, the notation being as in (3.3), where the intervals H in $[-\frac{1}{2}, \frac{1}{2}[$ have rational endpoints. The family of all such intervals and the family of all cylinders in $\Delta_{\mathbf{a}}$ are both countably infinite. Accordingly our open set U is a countable union of intervals in $\Sigma_{\mathbf{a}}$:

$$U = \bigcup_{n=1}^{\infty} H_n \times C_n. \tag{1}$$

We next write

$$\begin{aligned} U &= (H_1 \times C_1) \cup \bigcup_{n=2}^{\infty} \Big((H_n \times C_n) \setminus \bigcup_{k=1}^{n-1} (H_k \times C_k) \Big) \\ &= (H_1 \times C_1) \cup \bigcup_{n=2}^{\infty} \Big((H_n \times C_n) \cap \bigcap_{k=1}^{n-1} (H_k \times C_k)' \Big) \\ &= J_1 \cup \bigcup_{n=2}^{\infty} J_n. \end{aligned} \tag{2}$$

It is obvious that the intersection of two intervals in $\Sigma_{\mathbf{a}}$ is an interval in $\Sigma_{\mathbf{a}}$ or is void. It is a simple matter to show that the complement of an interval in $\Sigma_{\mathbf{a}}$ is a finite union of pairwise disjoint intervals in $\Sigma_{\mathbf{a}}$: we omit the details. Applying these observations to the sets J_n in (2), we see that each J_n is a finite union of pairwise disjoint intervals in $\Sigma_{\mathbf{a}}$. The sets J_n are constructed so as to be pairwise disjoint. This completes the proof.

(3.5) **Theorem.** *Let S be an arbitrary subset of $\Sigma_{\mathbf{a}}$ and ε an arbitrary positive number. There is a countable family $\{I_m\}_{m=1}^{\infty}$ of pairwise disjoint intervals in $\Sigma_{\mathbf{a}}$ such that*

(i) $S \subset \bigcup_{m=1}^{\infty} I_m$

and

(ii) $\mu(S) > \sum_{m=1}^{\infty} \mu(I_m) - \varepsilon$.

Proof. The general theory of measure shows that

$$\mu(S) = \inf\{\mu(U) : U \text{ is open and } U \supset S\}.$$

See for example [10], Theorem (11.22). Choose such a set U for which

$$\mu(S) > \mu(U) - \varepsilon. \tag{1}$$

Use Theorem (3.4) to write $U = \bigcup_{m=1}^{\infty} I_m$, where the sets I_m are pairwise disjoint intervals. Since intervals are Borel subsets of $\Sigma_{\mathbf{a}}$, every Borel set is μ–meaurable, and μ is countably additive on μ–measurable sets, (1) shows that

$$\mu(S) > \mu\Big(\bigcup_{m=1}^{\infty} I_m \Big) - \varepsilon = \sum_{m=1}^{\infty} \mu(I_m) - \varepsilon. \qquad \square$$

4. Divergent Fourier Series of Continuous Functions on $\Sigma_{\mathbf{a}}$

We will prove the following fact.

(4.1) **Theorem.** *Let N be any subset of $\Sigma_{\mathbf{a}}$ with Haar measure zero. There is a function $f \in \mathfrak{C}(\Sigma_{\mathbf{a}})$ such that the sequence $(S_n f(t, \mathbf{x}))_{n=0}^{\infty}$ is unbounded at every point $(t, \mathbf{x}) \in N$.*

For our proof we need a lemma of Kahane and Katznelson [15], which we state in a slightly sharpened form.

(4.2) **Lemma.** *Let E be a subset of $\mathbb{T}$ that is the union of a finite number of closed intervals, not necessarily disjoint, and suppose that $\lambda(E) > 0$. Let α be any number greater than 1. There is a polynomial Φ on $\mathbb{T}$ such that:*

(i) $\Phi(\exp(2\pi it)) = \sum_{n=0}^{2M} a_n \exp(2\pi int)$;

(ii) $\|\Phi\|_\infty < 1$;

(iii) $\left|\sum_{n=0}^{M-1} a_n \exp(2\pi int)\right| > \frac{1}{\pi}\log\left(\frac{1}{\alpha\lambda(E)}\right)$

for all $\exp(2\pi it)$ *in E.*

Proof. Repeat the proof in [16], pp. 57–58, noting that the function ψ_I can be chosen so that its real part exceeds $\frac{1}{\alpha\varepsilon}$ on I, and replacing e^{-iMt} by e^{iMt} in the definition of the function ϕ. □

(4.3) *Proof of* (4.1). Since $\mu(N) = 0$, we may apply Theorem (3.5) to find, for every positive integer l, a family $\{I_{l,n}\}_{n=0}^\infty$ of pairwise disjoint intervals in $\Sigma_{\mathbf{a}}$ such that

$$\sum_{n=1}^\infty \mu(I_{l,n}) \le \frac{1}{e}2^{-l} \tag{1}$$

and

$$N \subset \bigcup_{n=1}^\infty I_{l,n}. \tag{2}$$

It is obvious that

$$\varrho = \sum_{l=1}^\infty \left(\sum_{n=1}^\infty \mu(I_{l,n}\right) \le \frac{1}{e}. \tag{3}$$

For every positive integer k, there is a positiv integer n_k such that

$$\varrho - \sum_{l=1}^{n_k}\sum_{n=1}^{n_k} \mu(I_{l,n}) \le \exp(-(k+1)^3). \tag{4}$$

Without loss of generality, we suppose that $n_1 < n_2 < n_3 < \dots$. Write

$$M_1 = \bigcup_{l=1}^{n_1}\bigcup_{n=1}^{n_1} I_{l,n}, \tag{5}$$

and

$$M_k = \Big(\bigcup_{l=n_{k-1}+1}^{n_k}\ \bigcup_{n=1}^{n_k} I_{l,n}\Big) \cup \Big(\bigcup_{l=1}^{n_k}\ \bigcup_{n=n_{k-1}+1}^{n_k} I_{l,n}\Big) \quad \text{for } k = 2, 3, \dots\ . \tag{6}$$

It is plain that

$$\mu(M_1) \le \varrho \le \frac{1}{e}.$$

The set of M_k is contained in $\bigcup I_{l,n}$, the union being taken over the set $A(k)$ consisting of all pairs of indices (l, n) with $\max(l, n) > n_{k-1}$. From (4) we see that

$$\mu(M_k) \le \sum_{(l,n)\in A(k)} \mu(I_{l,n}) = \varrho - \sum_{l=1}^{n_{k-1}}\sum_{n=1}^{n_{k-1}} \mu(I_{l,n}) \le \exp(-k^3). \tag{7}$$

The relations (2), (5), and (6) imply that

$$N \subset \bigcap_{j=1}^{\infty} \bigcup_{h=j}^{\infty} M_h = \limsup_{h\to\infty} M_h. \tag{8}$$

Definitions (5) and (6) show that each set M_k is a finite union of intervals in the group $\Sigma_{\mathbf{a}}$. We may take these intervals to be pairwise disjoint, as in the proof of (3.4). We write

$$M_k = \bigcup_{l=1}^{s_k} H_l \times C(F_l, \mathbf{c}_l). \tag{9}$$

With no loss of generality, we may suppose that the sets F_l are all equal to $\{0, 1, 2, \ldots, r_k - 1\}$ for a positive integer r_k that depends only upon k. Possibly each H_l consists of a single point. In this case, enlarge some H_l to be an interval in $[-\frac{1}{2}, \frac{1}{2}[$ of positive length such that $\mu(M_k)$ remains less than $\exp(-k^3)$. We now consider the set

$$\pi_{r_k}(M_k) = P_k \subset \mathbb{T}.$$

It is clear that

$$P_k = \bigcup_{l=1}^{s_k} \pi_{r_k}(H_l \times C(F_l, \mathbf{c}_l)).$$

Let $\beta(\mathbf{c}_l)$ be the complex number

$$\exp\Big[2\pi i\Big(\frac{1}{A_{r_k}}\mathbf{c}_l(0) + \frac{a_0}{A_{r_k}}\mathbf{c}_l(1) + \cdots + \frac{1}{a_{r_k-1}}\mathbf{c}_l(r_k - 1)\Big)\Big].$$

It follows from the definition of π_{r_k} ((3.1.3) and (3.1.4)) that $\pi_{r_k}(H_l \times C(F_l, \mathbf{c}))$ is the set of all numbers

$$\Big\{\beta(\mathbf{c}_l)\exp\Big[2\pi i\Big(\frac{t}{A_{r_k}}\Big)\Big] : t \in H_l\Big\}. \tag{10}$$

This set is an interval in $\mathbb{T}$ whose Lebesgue measure is $\frac{1}{A_{r_k}}\lambda(H_l)$. (This set is a single point if H_l is a single point.) In any case, P_k is the union of a finite number of intervals in $\mathbb{T}$. The relations

$$0 < \lambda(P_k) = \mu(M_k) \tag{11}$$

are obvious from (9), (10), and the fact that π_{r_k} can be regarded as a one–to–one mapping of $\Sigma_{\mathbf{a}}/\Lambda_{r_k}$ onto $\mathbb{T}$. Note that $M_k = M_k + \Lambda_{r_k}$.

We now apply the lemma of Kahane and Katznelson (4.2) to the subset P_k of $\mathbb{T}$. We find a trigonometric polynomial

$$\Phi_k = \Phi_k(\exp[2\pi it]) = \sum_{l=1}^{u_k} a_{l,k}\exp[2\pi ilt] \tag{12}$$

on $\mathbb{T}$ such that:

$$\|\Phi_k\| < 1; \tag{13}$$

there is a $v_k < u_k$ such that

$$\Big|\sum_{l=0}^{v_k} a_{l,k}\exp[2\pi ilt]\Big| > \frac{1}{\pi}\log\Big(\frac{1}{\alpha\lambda(P_k)}\Big) = \frac{1}{\pi}\log\Big(\frac{1}{\alpha\mu(M_k)}\Big) \tag{14}$$

for all $\exp[2\pi it]$ in P_k. Let us define

$$\phi_k = \Phi_k \circ \pi_{r_k}. \tag{15}$$

That is,

$$\phi_k(t,\mathbf{x}) = \sum_{l=0}^{u_k} a_{l,k} \exp\left[2\pi i\Big(\frac{1}{A_{r_k}}\Big(t+\sum_{j=0}^{\infty} A_j x_j\Big)\Big)\right] > \sum_{l=0}^{u_k} a_{l,k}\chi_{l/A_{r_k}}(t,\mathbf{x}) \tag{16}$$

for all $(t,x) \in \Sigma_{\mathbf{a}}$.

From (13) and (15), we have

$$\|\phi_k\| < 1. \tag{17}$$

Since $\overset{-1}{\pi}_{rk}(P_k) = \overset{-1}{\pi}_{rk}(\pi_{rk}(M_k)) = M_k$, (14) and (15) show that

$$\Big|\sum_{l=0}^{v_k} a_{l,k} \exp\left[2\pi i\Big(\frac{1}{A_{r_k}}\Big(t+\sum_{j=0}^{\infty} A_j x_j\Big)\Big)\right]\Big| > \frac{1}{\pi}\log\Big(\frac{1}{\alpha\mu(M_k)}\Big) \tag{18}$$

for all $(t,\mathbf{x}) \in M_k$.

We now define by induction a sequence $(j_k)_{k=1}^{\infty}$ in $\mathbb{N}$ and a sequence $(\alpha_k)_{k=1}^{\infty}$ of positive numbers in the group $\mathbb{Q}_{\mathbf{a}}$. Let j_1 be a positive integer (the smallest one if you like) such that

$$r_1 < j_1 \tag{19}$$

and

$$\frac{v_1}{A_{r_1}} \le \frac{c_{j_1}-1}{A_{j_1}}. \tag{20}$$

We can find j_1 in view of (2.1.1) and (2.1.2). Now let b_1 be the nonnegative integer such that

$$\frac{v_1+b_1}{A_{r_1}} \le \frac{c_{j_1}-1}{A_{j_1}} < \frac{v_1+b_1+1}{A_{r_1}} \tag{21}$$

and let

$$\alpha_1 = \frac{b_1}{A_{r_1}} + \frac{1}{A_{j_1}}. \tag{22}$$

Consider the possible frequencies of the polynomial $\chi_{\alpha_1}\phi_1$: they are contained in the set

$$\Big\{\frac{b_1}{A_{r_1}}+\frac{1}{A_{j_1}}, \frac{1+b_1}{A_{r_1}}+\frac{1}{A_{j_1}}, \dots, \frac{v_1+b_1}{A_{r_1}}+\frac{1}{A_{j_1}}\Big\} \cup \Big\{\frac{v_1+b_1+1}{A_{r_1}}+\frac{1}{A_{j_1}}, \dots, \frac{u_1+b_1}{A_{r_1}}+\frac{1}{A_{j_1}}\Big\}$$
$$= E_{1,1} \cup E_{1,2}.$$

Next consider the complete blocks C_j and first entrance blocks B_j defined in (2.1.3) and (2.1.4). A trifling calculation based on (19) shows that $(E_{1,1}\cup E_{1,2}) \cap C_{j_1-1}$ is void, while (21) shows that $(E_{1,1}\cup E_{1,2}) \cap B_{j_1} = E_{1,1}$ and $E_{1,2} \cup C_{j_1}$ is void. Thus we have

$$S_{j_1}(\chi_{\alpha_1}\phi_1) = \chi_{\alpha_1}\sum_{l=0}^{v_1} a_{l,1}\chi_{l/A_{r_1}}. \tag{23}$$

We proceed by induction. Suppose that $j_1, j_2, \dots, j_{k-1}$ and $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$ have been defined. We select the positive integer j_k so that:

$$r_k < j_k; \tag{24}$$

the complete block C_{j_k-1} contains all of the frequencies of the polynomials $\chi_{\alpha_1}\phi_1, \chi_{\alpha_2}\phi_2, \ldots, \chi_{\alpha_{k-1}}\phi_{k-1}$; and

$$\frac{v_k}{A_{r_k}} \leq \frac{c_{j_k}-1}{A_{j_k}} \tag{25}$$

Let b_k be the nonnegative integer such that

$$\frac{v_k+b_k}{A_{r_k}} \leq \frac{c_{j_k}-1}{A_{j_k}} < \frac{v_k+b_k+1}{A_{r_k}}, \tag{26}$$

and define

$$\alpha_k = \frac{b_k}{A_{r_k}} + \frac{1}{A_{j_k}}.$$

The frequencies of the polynomial $\chi_{\alpha_k}\phi_k$ are contained in the set

$$\Big\{\frac{b_k}{A_{r_k}} + \frac{1}{A_{j_k}}, \frac{1+b_k}{A_{r_k}} + \frac{1}{A_{j_k}}, \ldots, \frac{v_k+b_k}{A_{r_k}} + \frac{1}{A_{j_k}}\Big\} \cup \Big\{\frac{v_k+b_k+1}{A_{r_k}} + \frac{1}{A_{j_k}}, \ldots, \frac{u_k+b_k}{A_{r_k}} + \frac{1}{A_{j_k}}\Big\}$$
$$= E_{k,1} \cup E_{k,2}.$$

The relation (24) shows that $(E_{k,1} \cup E_{k,2}) \cap C_{j_k-1}$ is void and (26) that $E_{k,1} \subset B_{j_k}$ and $E_{k,2} \cap C_{j_k}$ is void. Finally, we define a function f on $\Sigma_{\mathbf{a}}$ by

$$f = \sum_{h=1}^{\infty} \frac{1}{h^2}\chi_{\alpha_h}\phi_h. \tag{27}$$

From (17) we see that f is a continuous function: the series in (27) converges uniformly on $\Sigma_{\mathbf{a}}$. The uniform convergence of this series shows also that for every $\beta \in \mathbb{Q}_{\mathbf{a}}$,

$$\widehat{f}(\chi_\beta) = \sum_{h=1}^{\infty} \frac{1}{h^2}\widehat{\chi_{\alpha_h}\phi_h}(\chi_\beta).$$

The number $(\chi_{\alpha_h}\phi_h)\widehat{\ }(\chi_\beta)$ is the coefficient of χ_β in the polynomial $\chi_{\alpha_h}\phi_h$ if β is a frequency of $\chi_{\alpha_h}\phi_h$ and is 0 otherwise. Furthermore, our construction ensures that all of the rational numbers β lying in a given C_{j_k} that are frequencies of any polynomial $\chi_{\alpha_h}\phi_h$ occur in the sum $\sum_{h=1}^{k}\frac{1}{h^2}\chi_{\alpha_h}\phi_h$. We have proved that

$$|S_{j_k}f - S_{j_{k-1}}f| = \frac{1}{k^2}\Big|\chi_{\alpha_k}\sum_{l=0}^{v_k} a_{l,k}\chi_{l/A_{r_k}}\Big|. \tag{28}$$

For $(t, \mathbf{x}) \in M_k$, use (28), (18), and (7) to show that

$$|S_{j_k}f(t,\mathbf{x}) - S_{j_{k-1}}f(t,\mathbf{x})| > \frac{1}{\pi k^2}\log\Big(\frac{1}{\alpha\mu(M_k)}\Big) \geq \frac{1}{\pi k^2}(k^3 - \log\alpha). \tag{29}$$

Finally, consider any point $(t, \mathbf{x}) \in N$. As noted in (8), $(t, \mathbf{x})$ is an infinite number of the sets M_k. From (28) and (29), we infer that

$$\lim_{k\to\infty} |S_{j_k}f(t,\mathbf{x}) - S_{j_{k-1}}f(t,\mathbf{x})| = \infty,$$

and so the sequence $(S_n f(t,\mathbf{x}))_{n=0}^{\infty}$ is unbounded. $\square$

5. An $\mathfrak{L}_1$ Fourier Series that Diverges Everywhere

(5.1) *Preliminaries.* We will establish an exact analogue of Kolmogorov's construction of an everywhere divergent $\mathfrak{L}_1$ Fourier series on $\mathbb{T}$. Most of our work is carried out on $\mathbb{T}$. Throughout (5.1)–(5.5), all functions are defined on $\mathbb{T}$, and $\mathfrak{L}_1$ norms and iniform norms refer to $\mathbb{T}$. For real–valued functions ϕ and ψ defined on $\mathbb{T}$, inequalities of the form $\phi < \psi$ and $\phi \leq \psi$ are to be interpreted as holding pointwise everywhere on $\mathbb{T}$.

For $f \in \mathfrak{L}_1(\mathbb{T})$, the Fourier coefficients $\widehat{f}(k)$ are defined as usual by

$$\widehat{f}(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\exp(2\pi it)) \exp(2\pi ikt) \, \mathrm{d}\lambda(t) \tag{1}$$

for all $k \in \mathbb{Z}$. For nonnegative integers n, we write also as usual

$$S_n f(\exp(2\pi it)) = \sum_{l=-n}^{n} \widehat{f}(l) \exp(2\pi ilt). \tag{2}$$

Let $p(\exp(2\pi it)) = \sum_{k=m}^{n} a_k \exp(2\pi ikt)$ with $a_m a_n \neq 0$ be a trigonometric polynomial on $\mathbb{T}$. The number $\max\{|m|, |n|\}$ is called the *degree of the polynomial* p.

The key to our construction is a theorem of Kahane [14], p. 105, Theorem 3, which reads as follows.

(5.2) **Theorem.** *Let $(\nu(k))_{k=1}^{\infty}$ be any nondecreasing sequence of positive integers such that $\lim_{k\to\infty} \nu(k) = \infty$. There exists a real–valued function f in $\mathfrak{L}_1(\mathbb{T})$ for which*

(i) $\sup_{1\leq r<\infty} S_{\nu(r)} f(\exp(2\pi it)) = \infty$

for λ–almost all t in $[-\frac{1}{2}, \frac{1}{2}[$.

We need to replace "λ–almost all" by "all" in Theorem (5.2). Throughout (5.3)–(5.5), the sequence ν will be as in (5.2).

(5.3) **Lemma.** *Let α, β, and δ be positive real numbers. There exists a real–valued trigonometric polynomial p on $\mathbb{T}$,*

(i) $p(\exp(2\pi it)) = \sum_{l=-N}^{N} a_l \exp(2\pi ilt) \qquad (a_{-l} = \bar{a}_l)$,

such that

(ii) $\|p\| < \frac{1}{2}\alpha$

and

(iii) $\lambda(\{\exp(2\pi it) : \max_{1\leq j\leq k} S_{\nu(j)} f(\exp(2\pi it)) > \beta\}) > 1 - \delta$.

Proof. Let f be as in (5.2). For each positive integer k, let A_k be the set

$$\left\{ \exp(2\pi it) : \max_{1\leq j\leq k} S_{\nu(j)} f(\exp(2\pi it)) > \frac{7\|f\|_1 \beta}{\alpha} \right\} \tag{1}$$

We have $A_1 \subset A_2 \subset \ldots$ and by (5.2.i), $\lambda\left(\bigcup_{r=1}^{\infty} A_r\right) = \lambda(\mathbb{T}) = 1$. Hence we can choose a positive integer k such that

$$\lambda(A_k) > 1 - \delta. \tag{2}$$

Let $V_{\nu(k)} = 2F_{2\nu(k)-1} - F_{\nu(k)}$ be the de la Vallée=Poussin kernel on $\mathbb{T}$, as defined for example in [16], p. 15 (F_j is the Fejér kernel of course). Let p be the trigonometric polynomial

$$p = \left(\frac{\alpha}{7\|f\|_1} f\right) * V_{\nu(k)}.$$

To check (ii) for the polynomial p, note that

$$\|p\|_1 \le \frac{\alpha}{7\|f\|_1}\|f\|_1\Big)\|V_{\nu(k)}\|_1 \le \frac{\alpha}{7}\cdot 3 < \alpha/2.$$

Since $\widehat{V}_{\nu(k)}(l) = 1$ for $|l| \le \nu(k)$, we also have

$$S_{\nu(j)}p = \frac{\alpha}{7\|f\|_1}S_{\nu(j)}f \tag{3}$$

for $j = 1, 2, \ldots, k$. The equalities (3) imply that the set described on the left side of (iii) contains A_k. Thus (2) proves (iii). □

(5.4) **Lemma**. *Let α and β be positive real numbers. There is a trigonometric polynomial q on $\mathbb{T}$ such that*

(i) $\|q\|_1 < \alpha$

and

(ii) $\max_{1\le r<\infty} |S_{\nu(r)}q| > \beta$.

Proof. We apply (5.3). For the number δ, take $\frac{1}{2}\exp\left(\frac{-4\beta\pi}{\alpha}\right)$. Let p be the polynomial of (5.3) for α and β as in the present lemma and for the δ just specified. The partial sums $S_{\nu(j)}p$ are real–valued trigonometric polynomials and so the sets

$$D_j = \{\exp(2\pi it) : S_{\nu(j)}p(\exp(2\pi it))| > \beta\}$$

are open sets that are finite unions of open intervals in $\mathbb{T}$. The same is true of the set

$$D = \{\exp(2\pi it) : \max_{1\le r<\infty} S_{\nu(r)}p(\exp(2\pi it))| > \beta\} \tag{1}$$

which is a finite union of sets D_j. (Since p is a polynomial, its Fourier series has only a finite number of distinct partial sums.) Let E be the set $\mathbb{T}\setminus D$. The set E is a finite union of closed intervals in $\mathbb{T}$, and so is either finite or has positive Lebesgue measure. By (5.3.iii), we have $\lambda(D) > 1-\delta$, and so $\lambda(E) = 1-\lambda(D) < \delta$. If E is finite, embed it in a finite union of closed intervals the sum of whose lengths is positive and less than δ. Let F denote the set E if E is infinite and the enlarged set if E is finite.

We now apply Lemma (4.2) to the set F, where the number α of (4.2) is taken as 2. With a trivial change, we find a trigonometric polynomial

$$\Phi(\exp(2\pi it)) = \sum_{n=0}^{2M} a_n \exp(2\pi int) \tag{2}$$

such that

$$\|\Phi\|_\infty < \alpha/2 \tag{3}$$

and

$$\Big|\sum_{n=0}^{M-1} a_n \exp(2\pi int)\Big| > \frac{\alpha}{2\pi}\log\frac{1}{2\lambda(F)} > 2\beta \tag{4}$$

for all points $\exp(2\pi it) \in F$.

Let γ be the degree of the polynomial p. Let k be any positive integer such that $\nu(k) > (M-1)+\gamma$. Let q_0 be the polynomial

$$q_0(\exp(2\pi it)) = \exp(2\pi i(\nu(k) - M + 1)t)\Phi(\exp(2\pi it)). \tag{5}$$

Finally let q be the polynomial

$$q = p + q_0 \tag{6}$$

For j such that $\nu(j) \leq \gamma + 1$, (5) shows that

$$S_{\nu(j)}q = S_{\nu(j)}p. \tag{7}$$

For $\exp(2\pi it)$ in D, (7) and (1) imply (ii).

Now consider a point $\exp(2\pi it)$ not in D. This point belongs to E and hence to F. Since it is in E, (1) implies that

$$\max_{1\leq r<\infty} |S_{\nu(r)}p(\exp(2\pi it))| \leq \beta. \tag{8}$$

We look at the individual partial sum $S_{\nu(k)}q(\exp(2\pi it))$, k being as above. We find

$$\begin{aligned} |S_{\nu(k)}q(\exp(2\pi it))| &= |S_{\nu(k)}p(\exp(2\pi it)) + S_{\nu(k)}q_0(\exp(2\pi it))| \\ &\geq |S_{\nu(k)}q_0(\exp(2\pi it))| - |S_{\nu(k)}p(\exp(2\pi it))| \\ &= A - B. \end{aligned} \tag{9}$$

Since $\exp(2\pi it)$ is in F, (4) and the definition (5) of q_0 show that

$$A = \Big|\sum_{n=0}^{M-1} a_n \exp(2\pi int)\Big| > 2\beta. \tag{10}$$

By (8), we have

$$B \leq \beta. \tag{11}$$

Combining (11), (10), and (9), we get (ii) for all points in E. This completes the proof. □

(5.5) **Theorem.** *There is a function f in $\mathfrak{L}_1(\mathbb{T})$ for which*

(i) $\sup_{1\leq r<\infty} |S_{\nu(r)}f(\exp(2\pi it))| = \infty$

for all $\exp(2\pi it) \in \mathbb{T}$.

Proof. We construct by induction a sequence of trigonometric polynomials $(q_l)_{l=1}^{\infty}$ on $\mathbb{T}$. We write γ_l for the degree of the polynomial q_l and k_l for the greatest positive integer s such that $\nu(s-1) < \gamma_l \leq \nu(s)$. We will apply Lemma (5.4) repeatedly. First let q_1 be as in (5.4) with $\alpha = \frac{1}{2}$ and $\beta = 2$:

$$\|q_1\|_1 < 2^{-1} \tag{1}$$

and

$$\max_{1\leq r<\infty} |S_{\nu(r)}q_1| > 2. \tag{2}$$

Suppose that $q_1, q_2, \ldots, q_m$ have been constructed. We apply Lemma (5.4) again to define q_{m+1} as a trignonometric polynomial such that

$$\|q_{m+1}\|_1 < 2^{-m-1} \min\{\nu(k_1)^{-1}, \nu(k_2)^{-1}, \ldots, \nu(k_m)^{-1}\} \tag{3}$$

and

$$\max_{1\leq r<\infty} |S_{\nu(r)}q_{m+1}| > 2\max\Big\{m+1, \Big\|\max_{1\leq r<\infty}\Big|S_{\nu(r)}\Big(\sum_{j=1}^{m} q_j\Big)\Big|\Big\|_\infty\Big\}. \tag{4}$$

Define the function f by

$$f = \sum_{j=1}^{\infty} q_j. \tag{5}$$

From (3) we see that f is in $\mathfrak{L}_1(\mathbb{T})$. Now fix a positive integer l. It is easy to verify that

$$S_{\nu(r)}f = S_{\nu(r)}\Big(\sum_{j=1}^{l-1} q_j\Big) + S_{\nu(r)}q_l + \sum_{j=l+1}^{\infty} S_{\nu(r)}q_j = \phi_1 + S_{\nu(r)}q_l + \phi_2. \tag{6}$$

For $j \geq l+1$ and $r \leq k_l$, (3) shows that

$$\|S_{\nu(r)}q_j\| \leq (2\nu(r)+1)\|q_j\|_1 < 2^{-j}\frac{2\nu(k_l)+1}{\nu(k_l)} \leq 3 \cdot 2^{-j}.$$

It follows that

$$\|\phi_2\|_\infty \leq 3 \sum_{j=l+1}^{\infty} 2^{-j} = 3 \cdot 2^{-l}. \tag{7}$$

It is clear that

$$\phi_1 \leq \max_{1 \leq r < \infty} \Big| S_{\nu(r)}\big(\sum_{j=1}^{l-1} q_j\big)\Big|\Big\|_\infty = c. \tag{8}$$

Combining (6), (7), and (8), we find for $r \leq k_l$

$$|S_{\nu(r)}f| > |S_{\nu(r)}q_l| - c - 3 \cdot 2^{-l}$$

and so

$$\sup_{1 \leq r < \infty} |S_{\nu(r)}f| \geq \max_{1 \leq r \leq k_l} |S_{\nu(r)}f| > \max_{1 \leq r \leq k_l} |S_{\nu(r)}q_l| - c - 3 \cdot 2^{-l}. \tag{9}$$

Since

$$\max_{1 \leq r < \infty} |S_{\nu(r)}q_l| = \max_{1 \leq r \leq k_l} |S_{\nu(r)}q_l|,$$

the relation (4) with $m+1$ replaced by l shows that

$$c < \frac{1}{2} \max_{1 \leq r \leq k_l} |S_{\nu(r)}q_l|.$$

Therefore (9) and a second application of (4) yield

$$\sup_{1 \leq r < \infty} |S_{\nu(r)}f| > \frac{1}{2} \sup_{1 \leq r < \infty} |S_{\nu(r)}q_l| - 3 \cdot 2^{-l} \geq l - 3 \cdot 2^{-l}. \tag{10}$$

Since l can be arbitrarily large, (10) implies (i). □

(5.6) **Theorem.** *There is a function F in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$ such that*

(i) $\sup_{1 \leq n < \infty} |S_n F(t, \mathbf{x})| = \infty$

for all $(t, \mathbf{x})$ in $\Sigma_{\mathbf{a}}$.

Proof. For all nonnegative integers j, let $\nu(j)$ be the integral part of the rational number c_j/A_j. Let f be the function on $\mathbb{T}$ described in Theorem (5.5), and let F be the function $f \circ \pi_0$. (All notation is as in Sects. 2 and 3.) Thus F is a function in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$. A simple calculation, which we omit, shows that the Fourier transform $\widehat{F}$ of F, which is a function on $\mathbb{Q}_{\mathbf{a}}$, has the form

$$\widehat{F}\Big(\frac{l}{A_k}\Big) = \begin{cases} \widehat{f}(l) & \text{for } k = 0 \text{ and } l \in \mathbb{Z}; \\ 0 & \text{for } k = 1, 2, 3, \ldots \text{ and } l \in \mathbb{Z}. \end{cases}$$

It follows that

$$S_n f(t,\mathbf{x}) = \sum_{\beta\in C_n} \widehat{F}(\beta)\chi_\beta(t,\mathbf{x}) = \sum_{l=-\nu_n}^{l=\nu_n} \widehat{f}(l)\chi_l(t,\mathbf{x}) = S_{\nu(n)} f(\exp(2\pi i t)). \tag{1}$$

The second line of (1) is unbounded over n for all $\exp(2\pi i t)$ in $\mathbb{T}$, as proved in (5.5). Hence (i) follows. $\square$

(5.7) *Remark.* The function F constructed in Theorem (5.6) is independent of the $\mathbf{x}$–coordinate in $\Sigma_{\mathbf{a}}$. We can also define F as $f\circ\pi_h$ for an arbitrary positive integer h. The essentials of the construction remain on $\mathbb{T}$ in any case.

6. $\mathfrak{L}_p$ Fourier Series on $\Sigma_{\mathbf{a}}$ Converge Almost Everywhere

Throughout this section, p denotes a fixed but arbitrary real number strictly greater than 1. The symbol C_p denotes a constant depending upon p, which may vary from one occurence to another. Combining the maximal theorem of Carleson–Hunt and a classical submartingale theorem of Doob, we will prove that $S_n f(t,\mathbf{x})$ converges almost everywhere on $\Sigma_{\mathbf{a}}$ to $f(t,\mathbf{x})$, for f in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$.

(6.1) *Notation.* Let n and k be nonnegative integers and f a function in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$. Let $T_{n,k}f$ be the trigonometric polynomial

$$T_{n,k}f = \sum \widehat{f}(s)\chi_s, \tag{1}$$

the sum being extended over all s in $\frac{1}{A_n}\mathbb{Z}$ such that $|s|\le \frac{c_k}{A_k}$. For each n and each $(t,\mathbf{x})$ in $\Sigma_{\mathbf{a}}$, let

$$N_n f(t,\mathbf{x}) = \sup\{|T_{n,0}f(t,\mathbf{x})|, |T_{n,1}f(t,\mathbf{x})|,\ldots,|T_{n,k}f(t,\mathbf{x})|,\ldots\}. \tag{2}$$

Let

$$Mf(t,\mathbf{x}) = \sup\{|S_0 f(t,\mathbf{x})|, |S_1 f(t,\mathbf{x})|,\ldots,|S_n f(t,\mathbf{x})|,\ldots\} \tag{3}$$

where $S_n f$ is as in (2.2.3).

(6.2) **Lemma.** *Let k and n be integers such that $0\le k<n$. Let D be any Borel subset of $\Sigma_{\mathbf{a}}$ that is the union of cosets of the subgroup Λ_k of $\Sigma_{\mathbf{a}}$. Let h/A_n be a number in $\mathbb{Q}_{\mathbf{a}}$ that cannot be written in the form h'/A_k. Then we have*

(i) $\int_D \chi_{h/A_n}(u)\,\mathrm{d}\mu(u) = 0$.

Proof. The hypothesis on D implies that $1_D = 1_D * \lambda_k$, and so we have

$$\begin{aligned}\int_D \chi_{h/A_n}(u)\,\mathrm{d}\mu(u) &= \int_{\Sigma_{\mathbf{a}}} 1_D*\lambda_k(u)\chi_{h/A_n}(u)\,\mathrm{d}\mu(u) = (1_D*\lambda_k)\widehat{\ }\Big(-\frac{h}{A_n}\Big)\\ &= \widehat{1_D}\Big(-\frac{h}{A_n}\Big)\widehat{\lambda_k}\Big(-\frac{h}{A_n}\Big) = \widehat{1_D}\Big(-\frac{h}{A_n}\Big)1_{(1/A_k)\mathbb{Z}}\Big(-\frac{h}{A_n}\Big) = 0.\end{aligned}$$ $\square$

(6.3) **Theorem.** *Let f be a real–valued function in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$. The sequence of functions $(N_0f, N_1f,\ldots,N_nf,\ldots)$ forms a submartingale. We also have*

(i) $\|N_nf\|_p \le C_p\|f\|_p$.

Proof. Let $\mathfrak{B}_n$ be the smallest σ–algebra of subsets of $\Sigma_{\mathbf{a}}$ with respect to which the character χ_{1/A_n} is measurable. This is the family of Borel subsets of $\Sigma_{\mathbf{a}}$ that are unions of cosets of the subgroup Λ_n. Let us for the moment fix the integer k. Let m and n be integers such that $0 \le m \le n$. Lemma (6.2) shows that

$$\int_D T_{n,k} f \, \mathrm{d}\mu = \int_D T_{m,k} f \, \mathrm{d}\mu \tag{1}$$

for all sets D in $\mathfrak{B}_m$. That is, the sequence of functions $(T_{n,k}f)_{n=0}^{\infty}$ is a martingale with respect to the sequence of σ–algebras $(\mathfrak{B}_n)_{n=0}^{\infty}$. (Note that all of the functions $T_{n,k}f$ are real–valued, since f is real–valued.) It is elementary to show that if (X_n) and (Y_n) are submartingales, then $(\max\{X_n, Y_n\})$ is also a submartingale. See for example [2], Theorem (57.5). Now let k run from 0 to a positive integer K. We find that the sequence of functions $(\max\{|T_{n,0}f|, \ldots, |T_{n,K}f|\})_{n=0}^{\infty}$ is a submartingale. Going to the limit we see that the sequence of functions $(N_n f)_{n=0}^{\infty}$ satisfies the submartingale inequalities. To show that $(N_n f)_{n=0}^{\infty}$ is a submartingale, we need only to prove that each function $N_n f$ is in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$. This of course will follow by Hölder's inequality from (i).

To complete the proof, we thus need only to establish (i). Fix n, and consider the function $f * \lambda_n$, for which it is clear that $(f * \lambda_n) * \lambda_n = f * \lambda_n$. We note the familiar inequalities

$$\|f * \lambda_n\|_p \le \|f\|_p \|\lambda_n\| = \|f\|_p. \tag{2}$$

Now let g be *any* function in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$ for which the equality $g * \lambda_n = g$ holds. Parametrizing the cosets of Λ_n as in (3.1.5) and using (3.1.3) we write

$$\int_{\Sigma_{\mathbf{a}}} g(t, \mathbf{x}) \, \mathrm{d}\mu(t, \mathbf{x}) = \int_{-1/2}^{1/2} g \circ \overset{-1}{\pi}_n(s) \, \mathrm{d}\lambda(s). \tag{3}$$

(This is a special case of the identity [11], (28.54.iii); it can be verified in the present case by starting with functions 1_A where $A = \overset{-1}{\pi}_n(I)$ and I is an interval in $[-\frac{1}{2}, \frac{1}{2}[$.) Note that $g \circ \overset{-1}{\pi}_n$ is a function on $\mathbb{T}$. From (3) and (2) we find that

$$\|(f * \lambda_n) \circ \overset{-1}{\pi}_n\|_p = \|f * \lambda_n\|_p \le \|f\|_p. \tag{4}$$

Let us find the Fourier coefficients of the function $\phi_n = (f * \lambda_n) \circ \overset{-1}{\pi}_n$. For $h \in \mathbb{Z}$, (3.1.3) and (3) give

$$\begin{aligned} \widehat{\phi}_n(h) &= \int_{-1/2}^{1/2} \exp[-2\pi i h s] \phi_n(s) \, \mathrm{d}\lambda(s) = \int_{\Sigma_{\mathbf{a}}} \exp[-2\pi i h \pi_n(t, \mathbf{x})] (f * \lambda_n)(t, \mathbf{x}) \, \mathrm{d}\mu(t, \mathbf{x}) \\ &= \int_{\Sigma_{\mathbf{a}}} \chi_{-h/A_n}(t, \mathbf{x}) f * \lambda_n(t, \mathbf{x}) \, \mathrm{d}\mu(t, \mathbf{x}) = (f * \lambda_n)\widehat{\ }\Big(\frac{h}{A_n}\Big). \end{aligned} \tag{5}$$

Note as well that

$$(f * \lambda_n)\widehat{\ } = \widehat{f \lambda_n} = \widehat{f} \, 1_{A_n^{-1}\mathbb{Z}}. \tag{6}$$

We may now apply a theorem of Carleson [5] and Hunt [12]. Let $M\phi_n$ be the function on $\mathbb{T}$ defined by

$$M\phi_n(s) = \sup_{m \ge 0} \Big| \sum_{l=-m}^{m} \widehat{\phi}_n(l) \exp[2\pi i l s] \Big|. \tag{7}$$

That is, $M\phi_n$ is the classical maximal function for ϕ_n. Theorem 1 of Hunt [12] asserts that

$$\Big(\int_{-1/2}^{1/2} (M\phi_n(s))^p \, \mathrm{d}\lambda(s) \Big)^{1/p} \le C_p \|\phi_n\|_p. \tag{8}$$

We use (3) and (2) to rewrite (8) in the form

$$\Big(\int_{\Sigma_{\mathbf{a}}}(M\phi_n\circ\pi_n(t,\mathbf{x}))^p\,\mathrm{d}\mu(t,\mathbf{x})\Big)^{1/p}\leq C_p\|f\|_p. \tag{9}$$

The definition (6.1.1) of $T_{n,k}f$ makes it clear that

$$N_nf(t,\mathbf{x})\leq M\phi_n\circ\pi_n(t,\mathbf{x}) \tag{10}$$

for all $(t,\mathbf{x})$ in $\Sigma_{\mathbf{a}}$. From (9) and (10) we obtain

$$\Big(\int_{\Sigma_{\mathbf{a}}}(N_nf(t,\mathbf{x}))^p\,\mathrm{d}\mu(t,\mathbf{x})\Big)^{1/p}\leq C_p\|f\|_p.$$

This is (i). □

(6.4) **Theorem.** *For f in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$ we have*

(i) $\lim_{n\to\infty}\|S_nf-f\|_p{=}0$.

Proof. We use the notation employed in the proof of Theorem (6.3). A famous theorem of Riesz [19] (for a recent exposition, see [24], Ch. VII, p. 266, Theorem (6.4)) tells us that

$$\int_{-1/2}^{1/2}\Big|\sum_{h=-m}^{m}((f*\lambda_n)\circ\overset{-1}{\pi_n})\widehat{\ }(h)\exp[2\pi ihs]\Big|^p\,\mathrm{d}\lambda(s)\leq C_p^p\|(f*\lambda_n)\circ\overset{-1}{\pi_n}\|_p^p \tag{1}$$

for all $m\in\mathbb{Z}$. Set $m=c_n$ in (1) and recall that $S_nf=(S_nf)*\lambda_n$. Combining (1) and (6.3.3), we see that

$$\|S_nf\|_p\leq C_p\|f\|_p \tag{2}$$

for $n=0,1,2,\ldots$ Given a positive real number ε, write $f=f_1+f_2$, where f_1 is a trigonometric polynomial on $\Sigma_{\mathbf{a}}$ and $\|f_2\|<\varepsilon$. For all large enough n, the partial sum S_nf_1 is f_1 itself, and so (2) yields

$$\|S_nf-f\|_p\leq\|S_nf_1-f_1\|_p+\|S_nf_2\|_p+\|f_2\|_p\leq(C_p+1)\|f_2\|_p<(C_p+1)\varepsilon.$$

This of course implies (i) □[†]

(6.5) **Theorem.** *For all $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$, we have:*

(i) *the maximal function Mf is in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$ and $\|Mf\|_p\leq C_p\|f\|_p$;*

(ii) *the sequence $(S_nf(t,\mathbf{x}))_{n=0}^{\infty}$ converges to $f(t,\mathbf{x})$ for μ–almost all $(t,\mathbf{x})$ in $\Sigma_{\mathbf{a}}$.*

Proof. We apply Doob's theorem (Ch. VII (3.4), p. 317 of the treatise [6]) to the submartingale $(N_n)_{n=0}^{\infty}$. This yields

$$\int_{\Sigma_{\mathbf{a}}}\max\{N_0^p,N_1^p,\ldots,N_m^p\}\,\mathrm{d}\mu\leq\Big(\frac{p}{p-1}\Big)^p\int_{\Sigma_{\mathbf{a}}}N_m^p\,\mathrm{d}\mu, \tag{1}$$

for all nonnegative integers m. Apply (6.3.i) to the right side of (1) and B. Levi's theorem to the left. We find

$$\int_{\Sigma_{\mathbf{a}}}\sup\{N_0^p,N_1^p,N_2^p\ldots\}\,\mathrm{d}\mu\leq C_p^p\|f\|_p^p. \tag{2}$$

[†]Dr. Walter Bloom has also proved this theorem (written communication)

Since $S_n f$ is the trigonometric polynomial $T_{n,n}f$, the function $|S_n f|$ is majorized by $N_n f$. Therefore the inequality (2) proves (i).

With (i) and (6.4), it is easy to establish (ii). We sketch the argument. We have

$\int_{\Sigma_{\mathbf{a}}} M(f - S_n f)\,\mathrm{d}\mu \leq C_p \|f - S_n f\|_p = o(1),$

i.e.,

$\int_{\Sigma_{\mathbf{a}}} \sup_{m \geq n} |S_m f - S_n f|\,\mathrm{d}\mu = o(1).$

Therefore there is a subsequence $(n_k)_{k=0}^{\infty}$ of $\mathbb{N}$ such that

$$\lim\{\sup_{m \geq n_k} |S_m f(t,\mathbf{x}) - S_{n_k} f(t,\mathbf{x})|\} = 0 \tag{3}$$

for μ–almost all $(t,\mathbf{x})$ in $\Sigma_{\mathbf{a}}$. Since $\|S_{n_k} f - f\|_p = o(1)$, there is a subsequence $(n_{k_l})_{l=1}^{\infty}$ of $(n_k)_{k=1}^{\infty}$ such that

$$\lim_{l \to \infty} |S_{n_{k_l}} f(t,\mathbf{x}) - f(t,\mathbf{x})| = 0 \tag{4}$$

for μ–almost all $(t,\mathbf{x})$ in $\Sigma_{\mathbf{a}}$. Combine (3) and (4) to obtain (ii). □

(6.6) *Remarks.* It is known that the Fourier series of a function f on the circle $\mathbb{T}$ converges to f almost everywhere not only for f in $\mathfrak{L}_p(\mathbb{T})$ $(p > 1)$, but also if

$$\int_{\mathbb{T}} |f| \log^+ \log^+ |f|\,\mathrm{d}\mu < \infty.$$

See Sjölin [22] and for additional details Hunt and Taibleson [13], pp. 609–612. It is tempting to conjecture that a like result holds for functions on $\Sigma_{\mathbf{a}}$. In order to prove this, it would suffice to show that an inequality of the form

$$\mu(\{(t,\mathbf{x}) \in \Sigma_{\mathbf{a}} : Mf(t,\mathbf{x}) > y\})^{1/p} \leq \frac{C}{p-1}\|f\|_p \tag{1}$$

holds for all $y > 0$ and all f in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$, for $1 < p < 1 + \frac{1}{5\log 2}$. Details are given in [13], pp. 609–612. Our methods do not yield an inequality (1), but only (1) with $\frac{1}{p-1}$ replaced by a higher power of $\frac{1}{p-1}$. It would be interesting to establish (1) for the group $\Sigma_{\mathbf{a}}$, perhaps by a close study of Carleson's original construction, or by modifying Sjölin's proof to take account of (1) with $\frac{1}{(p-1)^{\alpha}}$.

(6.7) *Historical note.* After this paper has been written, Professor Kenneth A. Ross kindly drew our attention to the unpublished doctoral dissertation of Douglas N. Hawley [8]. Hawley introduced certain cases of our blocks C_j and used them to obtain analogues on $\Sigma_{\mathbf{a}}$ of Fejér and Poisson summability methods, which are of course classical for the group $\mathbb{T}$. To our knowledge, Hawley was the first to consider summing Fourier series on the groups $\Sigma_{\mathbf{a}}$ by using blocks of the form C_j.

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