A Uniform Hanani-Tutte Theorem for projective and radial planarity

Master Thesis of

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Statement of Authorship

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

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Abstract

A *drawing* $\mathcal{D}$ of a graph G is a representation of G. Such a drawing is called *planar*, if no pair of edges cross. Moreover, we can characterize a planar drawing of a graph by the circular ordering of the incident edges of each vertex, namely the *rotation system* of a drawing. If we have given a drawing of a graph with crossings, the different version of the *Hanani-Tutte theorems* are helpful to decide whether there is crossing-free drawing of this graph. If in the given drawing every pair of edges crosses an even number of times, we know by the *weak Hanani-Tutte theorem* , there is a crossing-free drawing of the graph, such that the rotation system is preserved. On the other hand, the *strong Hanani-Tutte theorem* provides that a graph has a planar drawing, if in the given drawing every pair of independent edges cross evenly. So, for this version, dependent edges can cross. The generalization of these two theorems is the *uniform Hanani-Tutte theorem*, where an independently even drawing of a graph is given. By the uniform theorem, there is a crossing-free drawing of that graph, such that the rotation system at even vertices is preserved. A vertex is even, if all incident edges cross all other edges evenly.

In this thesis, we consider the different versions of the Hanani-Tutte theorem in two settings. We start with the Hanani-Tutte theorems on the *projective plane.* Here, we present the proofs of both the weak and the strong Hanani-Tutte theorem on the projective plane. Afterwards, we show, why the representation of a graph in the proof of the strong version prevents the adaption to a proof of the uniform version. In the second part, we treat the Hanani-Tutte theorems for radial planarity. There we consider *level-graphs*, where each vertex corresponds to a level. A *radial drawing* of a graph, is then a drawing, where each vertex is assigned to a concentric cycle and the edges are drawn *radial.* That means the edges cross each level at most once. We present proofs for the weak and the strong Hanani-Tutte theorems for radial planarity. Then we prove the uniform Hanani-Tutte theorem for radial planarity under the assumption that odd crossings appear only at the extreme vertices v_1 or v_n. Moreover, we show, that a minimal counterexample to the theorem has to be connected and does not contain multiple edges.

Deutsche Zusammenfassung

Eine *Zeichnung* $\mathcal{D}$ eines Graphens G ist eine Darstellung von G. Eine solche Zeichnung ist *planar*, wenn sich keine Kantenpaare kreuzen. Außerdem können wir eine planare Zeichnung eines Graphen durch die zirkuläre Anordnung der Kanten jedes Knotens charakterisieren. Dies Anordnung bezeichnen wir als das *Rotationssystem* der Zeichnung. Wenn wir eine Zeichnung eines Graphen mit Kreuzungen gegeben haben, sind die verschiedenen Versionen der *Hanani-Tutte Theoreme* hilfreich um zu entscheiden, ob es eine kreuzungsfreie Zeichnung dieses Graphen gibt. Wenn sich in der gegebenen Zeichnung jedes Kantenpaar gerade oft kreuzt, wissen wir durch das *schwache Hanani-Tutte Theorem*, dass es eine kreuzungsfreie Zeichnung des Graphen gibt, in der das Rotationssystem erhalten bleibt. Wenn sich in der Zeichnung nur jedes Paar unabhängiger Kanten gerade oft kreuzt, wissen wir durch das *starke Hanani-Tutte Theorem*, dass eine planar Zeichnung existiert, in der jedoch das Rotationssystem nicht erhalten bleibt. In dieser Version können sich dafür abhängige Kanten kreuzen. Die Verallgemeinerung dieser Theoreme stellt das *uniforme Hanani-Tutte Theorem* dar. Bei diesem ist wiederum ein Zeichnung eines Graphen gegeben, in der sich alle unabhängigen Kanten gerade oft kreuzen. Nach dem uniformen Theorem gibt es dann eine kreuzungsfreie Zeichnung dieses Graphen, sodass das Rotationssystem an *geraden* Knoten erhalten bleibt. Das sind Knoten, bei denen alle inzidenten Kanten jede andere Kante gerade oft kreuzen.

Wir betrachten die verschiedenen Versionen der Hanani-Tutte Theoreme in zwei Varianten. Wir beginnen mit den Hanani-Tutte Theoremen für die *projektiven Ebene*. Wir präsentieren sowohl den Beweis des schwachen als auch des starken Hanani-Tutte Theorems. Danach zeigen wir, warum die Darstellung von Graphen im Beweis der starken Version die Anpassung des Beweises für einen Beweis der uniformen Version verhindert.
Im zweiten Teil behandeln wir die Hanani-Tutte Theoreme für radiale Planarität. Dabei betrachten wir *Level-Graphen*, bei denen jeder Knoten einem Level zugeordnet ist. Eine *radiale Zeichnung* eines Graphen ist dann eine Zeichnung, bei der jeder Knoten auf einem konzentrischen Kreis liegt und die Kanten *radial* gezeichnet sind. Das bedeutet, dass die Kanten jede Ebene höchstens einmal kreuzen. Wir stellen wiederum Beweise für das schwache und das starke Hanani-Tutte-Theorem für radiale Planarität vor. Dann beweisen wir das uniforme Hanani-Tutte Theorem für radiale Planarität unter der Annahme, dass ungerade Kreuzungen nur an den extremen Knoten v_1 und v_n auftreten. Außerdem zeigen wir, dass ein minimales Gegenbeispiel für das Theorem zusammenhängend sein muss und keine Mehrfachkanten enthalten darf.

Contents

1. Introduction

Given a graph $G = (V, E)$ with vertices and edges, it is often not quite clear whether we mean the graph itself or a drawing of the graph. Such a *drawing* is the arrangement of edges and vertices. But we can arrange the objects of a graph in many different way, which results in many different drawings of a graph. This can also result in drawings containing *crossings*, which are points in the drawing, where two edges intersect. Here, it is useful to find a drawing with as few crossings as possible. An example of different drawings of a graph with a varying number of crossings is shown in Figure 1.1. By the *crossing number* $\mathrm{cr}(G)$ of a graph G we can specify, how many crossings are at least necessary to draw the given graph. Hence, if $\mathrm{cr}(G) = 0$ then there exist a drawing of G without any crossing. We call such a drawing without crossings a *planar drawing* of G and the corresponding graph G *planar*.
However, if we now have a drawing of a graph G in which crossings are present, we can classify the vertices and edges of G. For example in Figure 1.1 (b) the edges e and f cross twice. So this pair of edges crosses evenly. Compared to this e and f cross in Figure 1.1 (c) only once, so they cross oddly. If a given edge e crosses every other edge of G an even number of times, we call e an *even edge*. If a graph G can be drawing only with even edges is represented by the *odd crossing number* $\mathrm{ocr}(G)$ of G, which specifies, how many pairs of edges have to cross at least an odd number of times in any drawing of G.
For a graph G, we would like to know if $\mathrm{cr}(G) = 0$, since then a planar drawing of G exists, which is often an easy-to-read representation. Let us assume for no, that we work with a

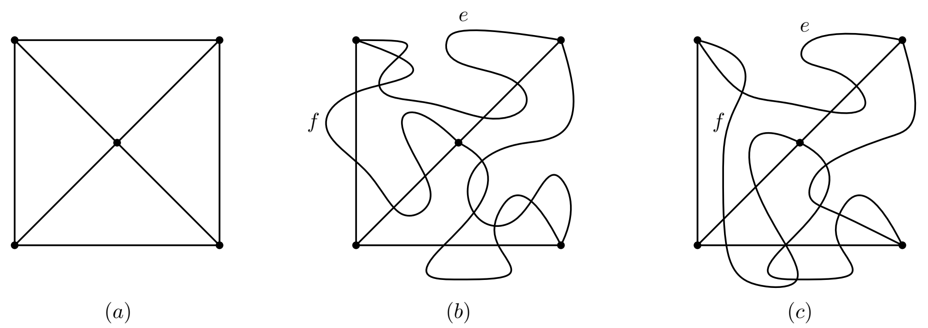


Figure 1.1: Three drawings of the same graph. (a) is planar drawing, (b) is a non-planar drawing, but all edges cross evenly and (c) is a non-planar drawing, where independent edges cross evenly, but dependent edges may cross oddly.

graph G with $\mathrm{ocr}(G) = 0$ and we are already given a drawing of G, where every edge is even. Then the Hanani-Tutte theorems are helpful, to establish a connection between a graph with $\mathrm{ocr} = 0$ and planarity. The Hanani-Tutte theorem states:

If there is for a given graph G a drawing realizing $\mathrm{ocr}(G) = 0$*, then there is a drawing realizing* $\mathrm{cr}(G) = 0$*.*

This means that a drawing that only contains even edges is just as good as a drawing that contains no crossings at all. To compare different drawings, a helpful tool are rotations. The *rotation* of a vertex v of G is the cyclic clockwise order of the edge incident to v in a drawing. The entirety of the rotations of all vertices of G together results in the *rotation system* of a drawing of G. By the rotation system of a planar drawing every face is determined. So if we again have a drawing of a graph G in which there are only even edges ($\mathrm{ocr}(G) = 0$), we also want to preserve the rotation scheme if possible. The *weak Hanani-Tutte theorem* does exactly this:

Theorem 1.1 (weak Hanani-Tutte theorem)**.** *Let $G = (V, E)$ be a graph given with a drawing realizing* $\mathrm{ocr}(G) = 0$*. Then there is a drawing realizing* $\mathrm{cr}(G) = 0$ *such that the rotation system is preserved.*

This is one of the actual version of the Hanani-Tutte theorems, by which a drawing with only even edges implies that there is also a crossing-free drawing of the same graph with the same rotation system. This theorem was proved independently multiple times. A geometric proof was done by Pelsmajer, Schaefer and Štefankovič [PSŠ07a]. Černỳ [Čer08] gave another proof in his PhD thesis. A proof that used homology theory and intersection forms was done by Cairns and Nikolayevsky [CN00]. This proof holds for arbitrary surfaces, not only for the plane.
So far we have only allowed even edges in the given drawings. But can we also allow some odd crossings, for example if the odd crossings only appear at pairs of *dependent* edges i.e. edges that have a common endpoint, see Figure 1.1 (*c*). If such a drawing exists for a given graph G is described by the *independent odd crossing number* $\mathrm{iocr}(G)$. This number specifies how many pairs of *independent* edges (edges not sharing an endpoint) cross at least in any drawing of G. For these drawings the *strong Hanani-Tutte theorem* establishes the connection to planar drawings:

Theorem 1.2 (strong Hanani-Tutte theorem)**.** *Let $G = (V, E)$ be a graph. If there is a drawing realizing* $\mathrm{iocr}(G) = 0$ *then there is a drawing realizing* $\mathrm{cr}(G) = 0$*.*

The strong version is the first stated version. It was proven by Hanani [Cho34] and Tutte [Tut70]. Note, that the naming is misleading. The strong version is not a generalization of the weak version, since the strong version makes no statement about the rotation system. Moreover, the weak version has the stronger result, and the strong version the weaker assumption. A common generalization is the *uniform Hanani-Tutte theorem.* This theorem combines the weak and strong Hanani-Tutte theorem in a natural way, by using the weaker assumption and maintaining the stronger result where possible. Hence, we assume that the given drawing is independently even, thus all pairs of independent edges cross an even number of times in the drawing. We want a planar drawing and that the rotation is preserved. But since there can exist odd crossings, the rotation system can nopt be preserved at every vertex. We can only preserve the rotation system at *even vertices*, that are vertices, where all incident edges are even.

Theorem 1.3 (uniform Hanani-Tutte theorem)**.** *Let $G = (V, E)$ be a graph given with a drawing realizing* $\mathrm{iocr}(G) = 0$*. Then there is a drawing realizing* $\mathrm{cr}(G) = 0$ *such that the rotation at even vertices is preserved.*

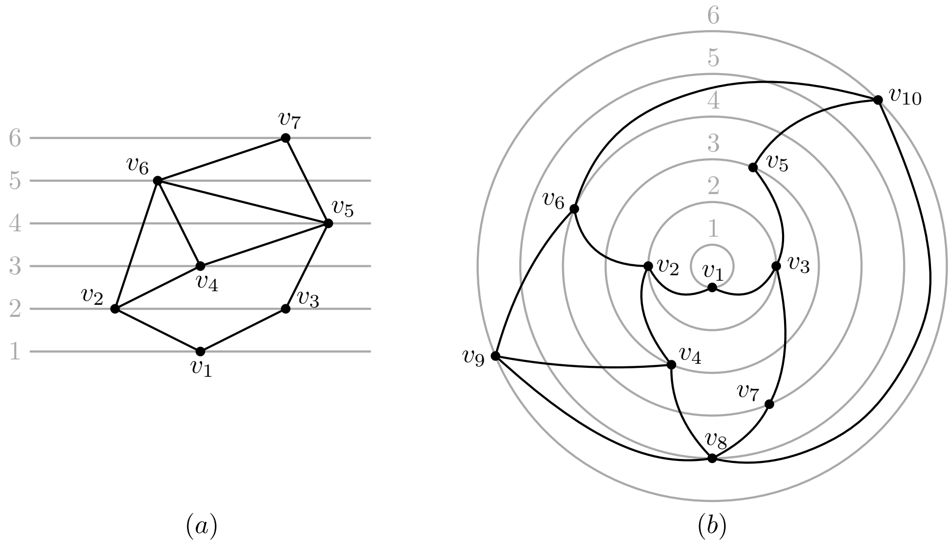


Figure 1.2: (a) A planar drawing of a level-graph. (b) A radial-planar drawing of a level-graph. Note these are two different graphs.

The uniform Hanani-Tutte theorem was shown by Fulek, Kynčl and Pálvölgyi [FKP17]. It was already indirectly stated by Pelsmajer, Schaefer and Štefankoviv̌ [PSŠ07a]. To prove the uniform version it is generally popular, to use the proof of the strong Hanani-Tutte theorem as a basis and adapt it accordingly. Note, that the uniform Hanani-Tutte theorem implies the weak and the strong Hanani-Tutte theorem. Therefore, if the strong Hanani-Tutte theorem is already wrong, also the uniform Hanani-Tutte theorem is disproved. More information about the Hanani-Tutte theorem in the plane can be found in "Hanani-Tutte and Related Results" by Schaefer [Sch13].

So far we only considered topological drawings in the plane. But there are also drawings in the plane with special properties. Assume, we have given a *level-graph.* For such a graph, the vertices are aligned on levels and the edges are monotone and connect vertices of different levels. An example of a level-graph is shown in Figure 1.2 (a). If the levels are not horizontal, but vertical, and the edges are monotone in x-direction, the representation is called an *x-monotone drawing.* For these restrictions, all three versions of the Hanani-Tutte theorems are proven [FPSŠ12, PT04, Bö22]. Thereby, the levels of the vertices are unchanged. A generalization of level-planarity is radial-planarity. Here the levels are represented by concentric cycles corresponding to the levels. This concept was firstly introduced by Bachmaier, Brandenburg and Forster [BBF05]. The weak and the strong Hanani-Tutte theorems for radial-planarity were proven by Fulek, Pelsmajer and Schaefer [FPS17, FPS23]. We will consider the uniform Hanani-Tutte theorem for radial planarity in Chapter 4.

There are also other surfaces then the plane, on which we can represent graphs. Let us start with the weak Hanani-Tutte theorem for any surface. There is a proof by Cairns and Nikolayevsky [CN00] using homology theory, that the weak Hanani-Tutte theorem holds on an orientable surface of any genus. A geometric proof by Pelsmajer, Schaefer and Štefanokovič [PSŠ07b] extends the result, such that the weak Hanani-Tutte theorem is also valid for non-orientable surfaces of any genus. By both proofs the weak Hanani-Tutte theorem is true for all surfaces of any genus.
For the strong Hanani-Tutte theorem, such a general prove does not exist and is generally incorrect [FK19]. But let us consider the strong Hanani-Tutte theorems for different surfaces step by step. We start with the non-orientable surfaces. There exists a proof of the strong Hanani-Tutte theorem on the projective plane. Pelsmajer, Schaefer and Stasi

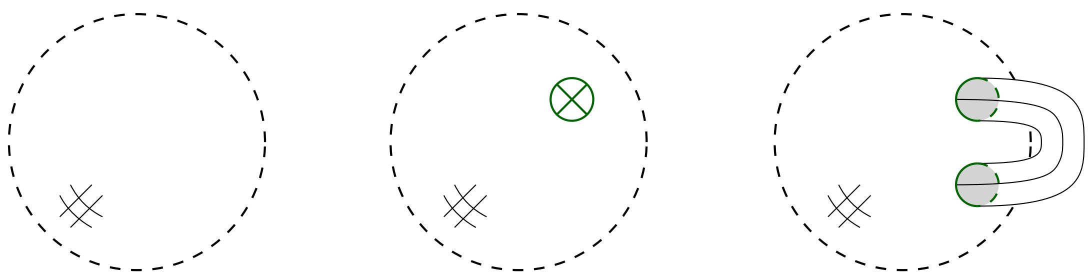
[PSS09] found a proof by using minimal forbidden minors for the projective plane. These minors were determined by Archedeacon, Glove, Huneke and Wang [Arc81, GHW79]. A direct proof for the strong Hanani-Tutte theorem on the projective plane was done later by Colin de Verdière et al. [CKP+17]. We will take a closer look at this proof in Chapter 3, where we also consider the uniform version. For all other non-orientable surfaces such results are known for neither the strong nor the uniform Hanani-Tutte theorem.
For orientable surfaces on the other hand, there exist some more results. We know by Fulek, Pelsmajer and Schaefer [FPS21] for the orientable surface of genus 1, namely the torus, that the strong Hanani-Tutte theorem is valid. For orientable surfaces of genus 2 or 3 we do not know is the strong or the uniform Hanani-Tutte theorems are true. For orientable surfaces of genus of at least 4 on the other hand there is a counterexample for the strong Hanani-Tutte theorem. To construct the counterexample Fulek and Kynčl [FK19] created a counterexample for the uniform Hanani-Tutte theorem on the torus first. Hence, neither the uniform Hanani-Tutte theorem on the torus nor the strong Hanani-Tutte theorem for orientable surfaces of genus of at least 4 hold. By the implication between the uniform and strong Hanani-Tutte theorems, the counterexample of Fulek and Kynčl [FK19] also implies that the uniform Hanani-Tutte theorems for orientable surfaces of genus of at least 4 are wrong.

Since the uniform Hanani-Tutte theorem for the torus is false, we go back to the plane, where we actually treat radial drawings. On the other hand, we will consider drawings on the non-orientable surface with lowest genus 1, namely the projective plane.
A common way to prove the uniform Hanani-Tutte theorem is to adapt the proof of the respective strong Hanani-Tutte theorem. So, we will present the strong proof in the two different settings and try to adapt it to an proof for the uniform version, respectively.
Hence, after introducing some preliminaries in Chapter 2, we treat the different version of the Hanani-Tutte theorem on the projective plane in Chapter 3. There, we start with the weak and strong Hanani-Tutte theorem on the projective plane. Building on this, we show, why we cannot develop the approach of the strong version further to a proof of the uniform Hanani-Tutte conjecture. Nevertheless, we are able to show a feature of a minimal counter example. The second main topic, namely the Hanani-Tutte theorem for radial planarity, is presented in Chapter 4. There we start with some topic-specific definitions, present the weak and strong Hanani-Tutte theorem for radial planarity and finally treat the uniform version.

2. Preliminaries

A *surface* is a connected compact Hausdorff topological space S that is locally homeomorphic to an open disk in the plane. It is also called *closed surface*. So, each point of S has an open neighborhood homeomorphic to the open unit disk in $\mathbb{R}^2$ [BM01]. Let B_1 and B_2 be two disjoint disks (with the same size) on a surface $\mathbb{S}$. By deleting the interior of B_1 and B_2 and identify the boundary ∂B_1 of B_1 with the boundary ∂B_2 of B_2 such that the clockwise orientations around ∂B_1 and ∂B_2 disagree, we obtain a *handle*. See Figure 2.1 (c) for an example of a handle on the sphere $\mathbb{S}_0$. To obtain a *crosscap* instead of a handle, we only add one disk B_1 to the surface $\mathbb{S}$. We delete the interior of B_1 and identify opposite points of the boundary ∂B_1. Figure 2.1 (b) sketches an example of a crosscap represented by $\otimes$ on $\mathbb{S}_0$.
To create different surfaces, we add a different number of handles or crosscaps to the sphere $\mathbb{S}_0$. By adding h handels to $\mathbb{S}_0$ we obtain the *orientable surfaces* $\mathbb{S}_h$. The simplest orientable surfaces are the sphere $\mathbb{S}_0$, the *torus* $\mathbb{S}_1$ and the *double torus* $\mathbb{S}_2$.
If we add k crosscaps instead of handles, we obtain *non-orientable surfaces* $\mathbb{N}_k$. By adding one crosscap to $\mathbb{S}_0$, we obtain the *projective plane* $\mathbb{N}_1$. By adding two crosscaps to the sphere, we get the *Klein Bottle* $\mathbb{N}_2$.
Given these definitions we are able to classify every surface, as stated by the following theorem.

Theorem 2.1 (Classification of Surfaces)**.** *Every surface is homeomorphic to precisely one of the surfaces $\mathbb{S}_h$ with $h \geq 0$, or $\mathbb{N}_k$ with $k \geq 1$.*



$\mathbb{S}_0$ $\mathbb{N}_1$ $\mathbb{S}_1$

Figure 2.1: On the left side there is a sketch of the sphere $\mathbb{S}_0$, in the middle figure the projective plane $\mathbb{N}_1$, where one crosscap (green) is added. On the right side there is sketch of the torus, where a handle is added to the sphere.

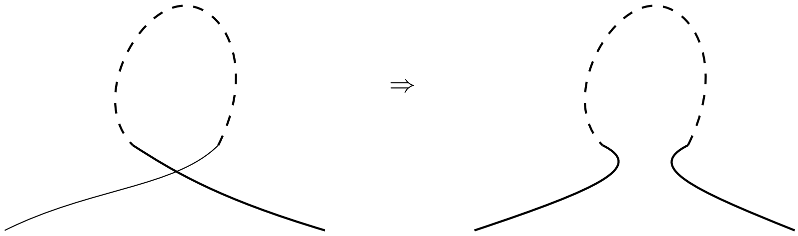


Figure 2.2: Sketch of the redrawing procedure to remove a self-intersection of an edge [PSŠ07b].

A proof of Theorem 2.1 was done by Thomassen [Tho92]. Hence, we assume to work with the described surfaces. For our purposes the sphere $\mathbb{S}_0$ and the projective plane $\mathbb{N}_1$ are enough. In Figure 2.1 (a) and b these two surfaces are sketched. In the following chapter we will represent the sphere by S^2 and the projective plane by $\mathbb{R}P^2$.

Let $\mathbb{S}$ be a surface and let $G = (V, E)$ be a graph with a vertex set V and an edge set E. A *drawing* $\mathcal{D}(G)$ of the graph G on a surface $\mathbb{S}$ is a representation of G such that each vertex $v \in V$ is drawn as a distinct point on $\mathbb{S}$. An edge $uv \in E$ is represented as a continuous curve on S connecting its endpoints u and v. A *crossing* is the point in a drawing, where two edges intersect. Thereby, we assume, that at one crossing point exactly two edges cross, and that two edges do not overlap, thus they intersect at finitely many crossings. Moreover, an edge does not pass over any vertex.
If in a given drawing edges have self-intersections, we can easily remove this intersection by redrawing the edge in a small neighbourhood around the crossing, as shown in Figure 2.2. This operation does not add any new crossings and only removes the self-intersection. Hence, we assume that the drawings we work with do not contain self-intersections.
We abuse the notation, such that a vertex v and an edge e represent both the objects of the graph G and their representation in the drawing $D(G)$.
If we consider the crossings between edges, it is enough for our purposes to check if two edges cross oddly or evenly, namely the *crossing parity* of the two edges. So, given a drawing $\mathcal{D}(G)$ of a graph G on a surface $\mathbb{S}$ and two edges e and f of G, the *number of crossings* between e and f modulo 2 is $\mathrm{cr}(e, f) = \mathrm{cr}_{\mathcal{D}}(e, f)$. Therefore, an edge e is *even*, if $\mathrm{cr}(e, f) = 0$ for every edge f different from e of G and *odd* otherwise. As a result of this definition, we do all calculations in $\mathbb{Z}_2$. We distinguish beyond that, if two edges cross *independently* even or odd. Then the two edges e and f must not share a common endpoint. If all edges of a graph are (independently) even, we call the graph *(independently) even*. Note, that there can be edges crossing oddly in an independently even drawing, but then the edges must be dependent. A vertex v in a drawing is *even*, if all its incident edges are even, otherwise v is an *odd* vertex.

A graph G is *planar*, if there exists a *planar drawing*, that is a drawing of G without any crossing. For example, the complete graph K_5 with five vertices is a non-planar graph on the plane, but on the projective plane it is a planer graph. Thus, if a graph is planar or non-planar depends on the considered surface, as shown in Figure 2.3. So, we want to find a planar drawing, also called an *embedding* $\mathcal{E}(G)$ of G on a given surface $\mathbb{S}$.
The *rotation* of a vertex v on any surface $\mathbb{S}$ is the cyclic clockwise order of all incident edges of v in the given drawing $\mathcal{D}$. The rotations of all vertices of G together are called the *rotation system* of G. This rotation system can be used to characterize an embedding of G, since by the given rotations all faces of G are determined. For each face, one can start at a vertex v and move along an edge vw of the face until we reach the vertex w. At w we pick the next edge to walk along by the rotation of w such that the new edge e is the successor of vw in the order. By repeating this operation, we obtain the hole face. So,

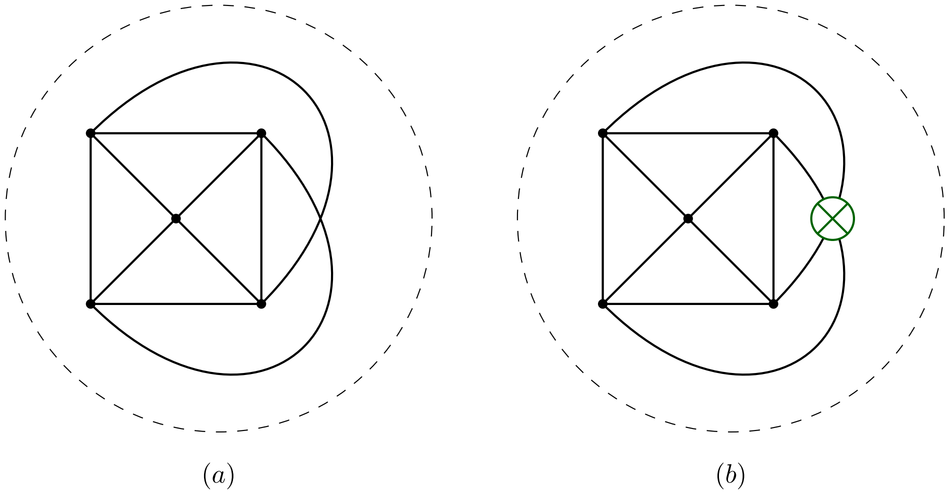


Figure 2.3: (a) An example of a non-planar graph (K_5) on the plane. (b) The same graph on the projective plane, where it is planar.

two embeddings are the same, if the rotation system is the same. That motivates, why we want to keep the rotation of a vertex.

A helpful concept will be the decomposition of a graph G into blocks. Such a *block* is either a maximal 2-connected subgraph with at least three vertices, an edge or an isolated vertex. Two different blocks of G intersect at most in one vertex by the maximality of these blocks. This vertex is called a *cutvertex*. Each edge of G is assigned to exactly one block by the decomposition of G into blocks [Die17, CKP$^+$17]. Given such a decomposition, we can work on the planarity of these blocks individually and get a planar graph by the following lemma.

Lemma 2.2. *Given a graph G with a decomposition into blocks. If all blocks are planar, then G is planar as well.*

Proof. Assume G is not planar. Then by Kuratowski‘s theorem G must contain K_5 or $K_{3,3}$ as a subgraph [Kur30]. Since the two graphs do not contain a cut vertex, one of the two must be contained in a single block. But then there is a block that is not planar. □

Given these preliminaries, we want to work on the different versions of the Hanani-Tutte theorems on the projective plane and for radial planarity.

3. Uniform Hanani-Tutte on the Projective Plane

In the following chapter, we want to discuss the Hanani-Tutte theorem for the projective plane. We start with the weak Hanani-Tutte theorem on the projective plane in Section 3.1. This section is based on the proof of Pelsmajer, Schaefer and Štefankovič [PSŠ07b]. Afterwards, we show in Section 3.2 the strong version. The proof in this section comes from Colin de Verdière et al. [CKP$^+$17]. The last section of this chapter will state some approaches about the uniform Hanani-Tutte conjecture on the projective plane.

3.1 Weak Hanani-Tutte on the Projective Plane

First we need an *embedding scheme* to characterize a drawing $\mathcal{D}$ of a graph G. This embedding scheme is the rotation system of $\mathcal{D}$ combined with a signature $\lambda : E \to \{-1, 1\}$. Since the projective plane is a non-orientable surface, we pick the rotation of a vertex clockwise with respect to one side of the surface near to the vertex. The signature of an edge $e = uv$ indicates if the rotations at u and v agree in the sense of clockwise rotation along e. If this is the case, the signature is 1 and otherwise -1.

Given a fixed embedding scheme of a drawing $\mathcal{D}$ of a graph G we will apply two main tools to change $\mathcal{D}$ into a planar drawing of G. The first one is removing self-intersections of an edge as defined in Chapter 2. Such self-intersections can occur during the process of making the drawing planar.

The second, more important, operation is the *contraction* of an edge $e = uv$. Thereby the vertex v is moved to u. For this, each edge f crossing e is moved over v such that f crosses all other edges that are incident to v. Since f is even, the crossing parity of these other edges is unchanged. Afterwards, we can combine u and v to obtain a new vertex u' as shown in Figure 3.1. The operation changes the embedding scheme naturally. Thus, if $\lambda(e) = 1$ the rotation of u' is the combined rotations of u and v such that the edges incident to v are integrated in the rotation of u in the local surrounding of e without intersecting the edges of u. The signature of the edges stays the same. If $\lambda(e) = -1$ we embed the edges of v in the same way, but we flip their signatures.

This tool is used in the following lemma, by which we are able to undo the contraction of an edge to get back the original setting.

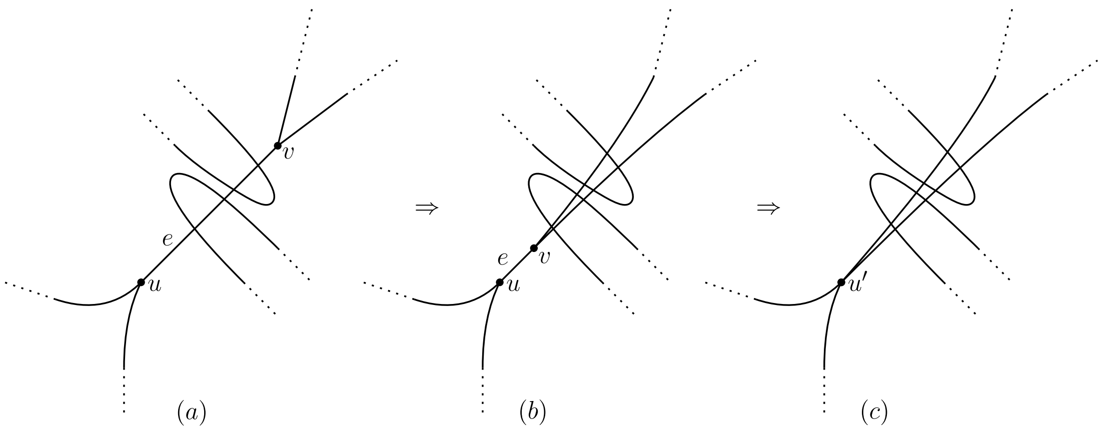


Figure 3.1: The steps of contracting an even edge $e = uv$ with resulting vertex u'.

Lemma 3.1. *Given a multigraph G together with an embedding scheme and a multigraph G' obtained from G by contracting an edge $e = uv$ with modified embedding scheme as given by the operation. Then, if G' can be embedded on the projective plane realizing its embedding scheme, also G can be embedded in the projective plane realizing its original embedding scheme.*

Proof. Let $\mathcal{E}'(G')$ be an embedding of G'. Now we undo the contraction. For this, we split the vertex u' back into the two vertices u and v. Thereby, the incident edges of u' are redistributed to u and v as they were in G. Then we connect u and v by an edge e such that no crossings are added. So, by use $\mathcal{E}'(G')$ for the rest of the graph, we obtain an embedding $\mathcal{E}(G)$. Moreover, if $\lambda(e)$ was 1 in G, we have to flip the values of lambda for all edges incident to v in $\mathcal{E}(G)$. □

Now we want to prove the weak Hanani-Tutte theorem for the projective plane. The following proposition is used in the proof of the theorem. Thereby, a *non-seperating curve* C is a curve on a surface S, such that if we remove C from S and obtain the surface $S - C$, $S - C$ is only one component. Otherwise C is a *separating curve.*

Proposition 3.2. *Let G be a multigraph with a single vertex v, drawn on a surface other than the sphere, so that all edges are even (loops). Then either G contains an edge e that is a non-separating curve C, or else we can draw a new non-seperating curve through v that crosses each edge of G an even number of times. (In the latter case, we can add a new edge e to G, drawn as the new curve, so that all edges are even.)*

Proof. Since we are on the projective plane, there is a non-seperating curve C. We assume that v is not on C. If C is even, we use C. We redraw a small segment of C such that the deformation does not push C over v, but C contains v afterwards between two consecutive edges in the rotation. Otherwise if C is not even, there must exist a loop e in G, that crosses C oddly. But then, since C and e are closed curves and they croos oddly, both are non-seperating. Hence, e is suitable curve. □

Theorem 3.3. *If G can be drawn on the projective plane so that all its edges are even, then G can be embedded on the projective plane, i.e. drawn crossing free, without changing the embedding scheme.*

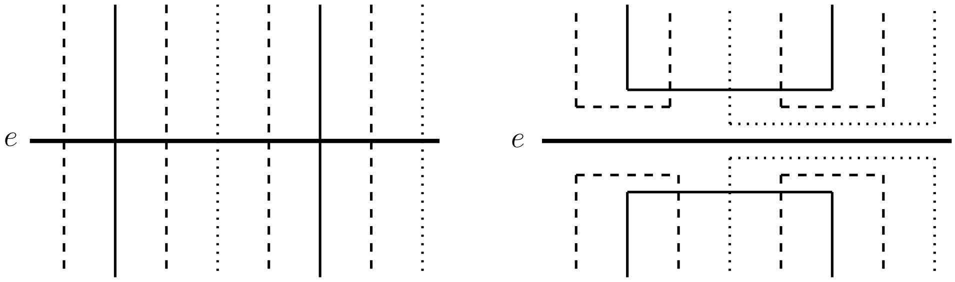


Figure 3.2: Elimination of even crossings of a edge e with three other edges.

Proof. We fix a drawing $\mathcal{D}(G)$ of a graph G on the projective plane. The proof is done by induction on the number of vertices of G. For this, we will use a slightly stronger statement:

If $\mathcal{D}$ is a drawing of a multigraph G on the projective plane such that every pair of edges crosses an even number of times, then G can be drawn without crossings on the projective plane without changing the embedding scheme.

If $\mathcal{D}$ contains an even edge uv with $u \neq v$, we use the contraction operation as shown in Figure 3.1. Since uv is an even edge, all edges which were incident to v stay even after the contraction operation. Then we get a new graph G' with a new vertex u' with joined rotations of u and v. Thereby, possible generated self-intersections can be removed by redrawing the crossing as shown in Figure 2.2. For G' exists a drawing on the projective plane without crossings and an unchanged embedding scheme by induction. Afterwards, we undo by Lemma 3.1 the contraction operation to get back the vertices u and v and the edge uv. Thus, we get a crossing free embedding of G on the projective plane with the same embedding scheme.
If G does not contain any contractable edge, we are left with one or more vertices. Assume we have at least two vertices u and v. Then the graph G is not connected. We introduce a new curve C from u to v which does not intersect any other edge. For this, we consider C as an even edge between u and v and can apply the contraction operation along C to get a new graph G' as in the previous case. By the induction hypothesis there is a crossing free drawing $\mathcal{D}'(G')$ of G' with unchanged embedding scheme. In $\mathcal{D}'(G')$ we can split the joined vertex back into u and v and ignore C. Then we get the desired drawing of G.
The last case is, that G is a single vertex u with loops. By Proposition 3.2 exists an edge e in G, that is a non-seperating curve or if no such edge is present, we can add such an edge e to G. The next step is to make e free of crossings. For this, we partition the crossings of e into pairs of crossings, such that each pair consists of two crossings between e and another but fixed edge f different from e. This is possible, since e is even, so there is an even number of crossings between e and any other edge f. We remove all such crossings by leaving e and cut the other edge f for each pair. This gives two loose ends of f on each side. These ends can be connected by an edge alongside of e. We perform this operation for all pairs of crossings in any particular order, as shown in Figure 3.2. Thereby, the crossing-parity of any pair of edges is not changed. Moreover, the operation produces "curves" with multiple components, where only one is connected to v. But these component can be glued back together. We consider the surface that is obtained from the projective plane by removing e. There we pick a "curve" g, that consists of multiple components. We deform a small part of a component of g such that it reaches another component of g. We then locally redraw the two component such that they form one component as shown in Figure 3.3. By avoiding u during the redrawing, the parity of crossings between any pair of edges stays even. Doing the described operation for all components, we get a drawing of G where each edge is a closed curve and e is crossing free.
To describe the rotation at a vertex, we transfer the cyclic order of the edges of the vertex

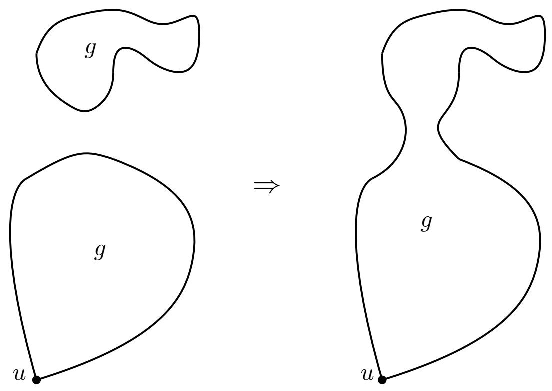


Figure 3.3: Redrawing a "curve" g with multiple components to get a curve consisting of just one component.

into a *word*, by representing each end of an edge by a letter. So, the rotation at u is described as ew_1ew_2 where w_1, w_2 are words containing the other edges of G.
We add a closed curve C through u along e to the drawing and remove e. By cutting the surface S along C, attaching a small disk to each component of $S - C$ and contract each disk to a point, we get a so called C-reduced surface S' with smaller genus than the projective plane. Thus, S' is a sphere. The cutting gives us a multigraph G' on two vertices u_1 and u_2. Thereby, the rotation at u_1 is w_1 and at u_2 is w_2.
Assume now, that C crosses the crosscap evenly. Then C is a two-sided curve, and we draw G' by the weak Hanani-Tutte theorem for the plane with the same embedding scheme. Afterwards we cut a hole at each u_i such that the hole touches u_i between the first and the last vertex of w_i. By glueing the holes boundaries together we get back G drawn on the projective plane with the original rotation scheme.
If C crosses the crosscap an odd number of times, C is called a one-sided curve. Then contracting the attached disk joins u_1 and u_2 back together to a new vertex u'. For this, the local rotation and the signature of incident edges at one vertex must be flipped. Let us pick u_1 to be this vertex. That yields to a rotation of $w_1w_2^R$ or $w_1^Rw_2$ at u', where R stands for "reversed". Since S' is orientable, we continue flipping local orientations and signatures until every signature is 1. The weak Hanani-Tutte theorem for the plane gives us a crossing free drawing of the new graph with the single vertex u'. Afterwards, we split u' back into u_1 and u_2 and cut a hole where u_1 and u_2 are on its boundary. Let C_1 be the part of the boundary of the hole from u_1 to u_2 and let C_2 be the other part. Identify C_1 and C_2 to recreate the original surface and the original graph G without crossings. □

This proof can be extended such that it holds for any non-orientable surfaces of any genus. Given the weak Hanani-Tutte theorem for the projective plane we continue with the strong version of the theorem on this surface.

3.2 Strong Hanani-Tutte on the Projective Plane

The goal of this section is to proof the following theorem, namely the strong Hanani-Tutte theorem for the projective plane.

Theorem 3.4. *A graph G can be embedded into the projective plane if and only if it admits a Hanani-Tutte drawing on the projective plane.*

Given an independently even drawing $\mathcal{D}(G)$ of a graph G on the projective plane. The idea of the proof is to find a *simple* cycle Z first. That means Z is even and free of self-intersections. Then Z splits the graph into an inside and an outside part. By duplicating the crosscap, such that at both sides there is one, we can find an embedding for G. We will afterwards show the main tool that is making one side planar without using the crosscap. So, we get an embedding of G on $\mathbb{R}P^2$, where only one crosscap is present.

To prove the strong Hanani-Tutte theorem on the projective plane we start by defining different types of drawings and their properties.

3.2.1 Hanani-Tutte Drawings

We start with a general definition of a (strong) Hanani-Tutte drawing.

Definition 3.5. *A drawing $\mathcal{D}(G)$ of a graph G on a surface S is a* (strong) Hanani-Tutte drawing *if every pair of independent edges cross an even number of times.*

In the following the term strong Hanani-Tutte drawing is usually abbreviated by *HT-drawing*. To talk about drawings on the projective plane, we define how to represent $\mathbb{R}P^2$. We take the sphere S^2 and a disk B. By removing the interior of B and identifying the opposite points of the boundary ∂B, we get a curve γ called *crosscap* that is represented by $\otimes$ in the figures. Thereby, γ is a homologically non-trivial simple cycle (loop) in $\mathbb{R}P^2$, and the other way round, every homologically non-trivial simple cycle (loop) can be used as a crosscap up to self-homomorphism of $\mathbb{R}P^2$. Moreover, we assume that an HT-drawing of a graph is crossing the crosscap only finitely often, otherwise we slightly shift the crosscap while keeping the properties of an HT-drawing.
To compare Hanani-Tutte drawings on S^2 and $\mathbb{R}P^2$, we introduce a map $\lambda : E(G) \to \mathbb{Z}_2$. This map assigns 0 to each edge e of a graph G if e and the crosscap γ cross an even number of times, and 1 if they cross oddly. Thereby, λ depends on the choice of γ. We later introduce vertex-crosscap switches, which alter λ. But λ allows checking whether a cycle Z is homologically nontrivial. Since Z is homologically nontrivial if and only if $\lambda(Z) = \sum_{e \in E(Z)} \lambda(e) = 1$ in $\mathbb{Z}_2$. This operation is independent of the choice of the crosscap. Given this map λ, we are able to define a projective Hanani-Tutte drawing on the sphere.

Definition 3.6. *Let D be a drawing of a graph G on S^2 and $\lambda : E(G) \to \mathbb{Z}_2$ be a function. Then the pair $(\mathcal{D}, \lambda)$ is a* projective HT-drawing *of G on S^2 if $\mathrm{cr}(e, f) = \lambda(e)\lambda(f)$ for any pair of independent edges e and f of G. (If λ is sufficiently clear from the context, we say that $\mathcal{D}$ is a projective HT-drawing of G on S^2.)*

The interesting part of this definition is, that we can transform a given HT-drawing $\mathcal{D}$ of a graph G on $\mathbb{R}P^2$ into a projective HT-drawing $\mathcal{D}'$ of G on S^2. It is enough, to remove the crosscap and let the edges crossing the crosscap cross at the position of the crosscap, as shown in the first two pictures of Figure 3.4. That is the easy direction, how to get from a projective HT-drawing on S^2 to an HT-drawing on $\mathbb{R}P^2$ is shown in the proof of the following lemma.

Lemma 3.7. *Let $(\mathcal{D}, \lambda)$ be a projective HT-Drawing of a graph G on S^2. Then there is an HT-drawing $\mathcal{D}_\otimes$ of G on $\mathbb{R}P^2$ such that $\mathrm{cr}_{\mathcal{D}_\otimes}(e, f) = \mathrm{cr}_{\mathcal{D}}(e, f) + \lambda(e)\lambda(f)$ for any pair of distinct edges of G, possibly adjacent. In addition, if e and f are two arbitrary edges such that $\lambda(e) = \lambda(f) = 0$ and $\mathcal{D}(e)$ and $\mathcal{D}(f)$ are disjoint; then $\mathcal{D}_\otimes(e)$ and $\mathcal{D}_\otimes(f)$ are disjoint as well.*

Proof. We insert a small disk B such that B and $\mathcal{D}$ do not intersect. As described above, we convert B into a crosscap γ. All edges e of G with $\lambda(e) = 1$ must be redrawn so that e passes through γ. For this, every such edge e is deformed towards the crosscap with

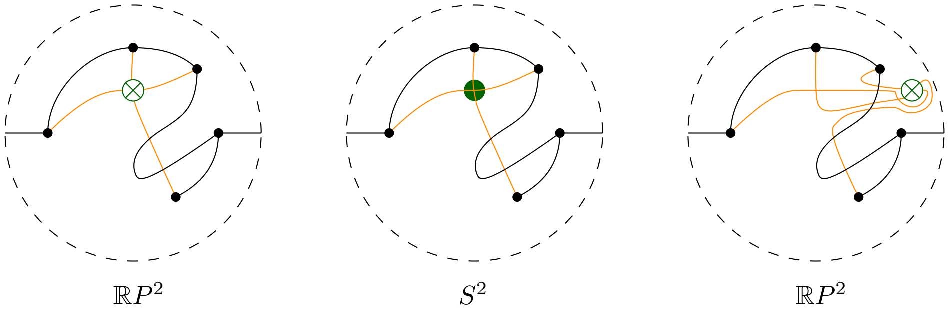


Figure 3.4: Relationship of HT-drawings on $\mathbb{R}P^2$ of a K_5 with a projective HT-drawing on S^2 of the same graph. The orange edges have a λ-value of 1.

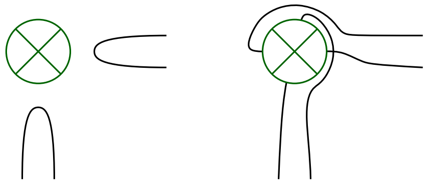


Figure 3.5: Redrawing the black edges around the crosscap with a finger-move.

a so called finger-move. We pull the edge to the crosscap, such that every other edge is intersected pairwise. Afterwards, e is drawn through γ as shown in Figure 3.5. This is done for all such edges in any order. Hence, all edges e with $\lambda(e) = 1$ go through γ and all edges e with $\lambda(e) = 0$ do not. This matches exactly the definition of λ for HT-drawings on $\mathbb{R}P^2$. Moreover, the redrawing around the crosscap adds a crossing between edges e and f that pass over the crosscap. So, $\mathrm{cr}_{\mathcal{D}_\otimes}(e, f) = \mathrm{cr}_{\mathcal{D}}(e, f) + \lambda(e)\lambda(f)$ holds for the new drawing $\mathcal{D}_\otimes$. An illustration of the hole process is shown in the second and third graphic of Figure 3.4. □

Hence, both directions are shown and Corollary 3.8 states, that we can switch between the representations and for projective HT-drawings on S^2 to actual position of the crosscap is irrelevant. This is helpful, if we want to merge two drawings. To differentiate between usual and projective HT-drawings on S^2 we call the usual one *ordinary HT-drawing*.

Corollary 3.8. *A graph G admits a projective HT-drawing on S^2 (with respect to some function $\lambda : E(G) \to \mathbb{Z}_2$) if and only if it admits an HT-drawing on $\mathbb{R}P^2$.*

Now, we want to define nontrivial walks, where we will often use special cases, namely edges, paths or cycles.

Definition 3.9. *Let $(\mathcal{D}, \lambda)$ be a projective HT-drawing of a graph G and let ω be a walk in G. Define $\lambda(\omega) := \sum_{e \in E(\omega)} \lambda(e)$ where $E(\omega)$ is the multiset of edges of ω. Then ω is* trivial*, if $\lambda(\omega) = 0$ and* nontrivial *otherwise.*

By this definition, a cycle Z is trivial if and only if Z is drawn as a homologically trivial cycle in $\mathcal{D}_\otimes$ of Lemma 3.7. Moreover, if there is a projective HT-drawing $(\mathcal{D}, \lambda)$ of a graph G with two such nontrivial cycles, then these cycles are not vertex-disjoint.

Lemma 3.10. *Let $(\mathcal{D}, \lambda)$ be a projective HT-drawing of a graph G on S^2. Then G does not contain two vertex-disjoint nontrivial cycles.*

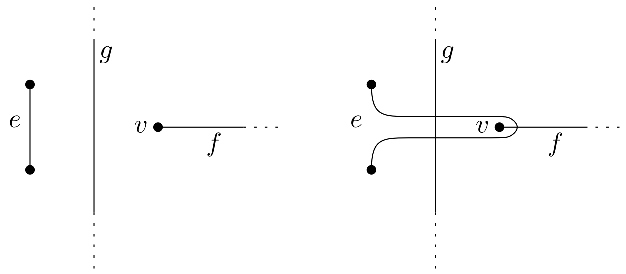


Figure 3.6: Execution of a vertex-edge switch (v, e). The crossing parity of e and f changes, while it stays the same between e and g.

Proof. We prove the lemma by contradiction. Let Z_1 and Z_2 be two vertex-disjoint nontrivial cycles of G. So, both cycles contain an odd number of nontrivial edges, such that $\lambda(Z_1) = \lambda(Z_2) = 1$. Hence, there exists an odd number of pairs (e_1, e_2) of nontrivial edges where $e_1 \in E(Z_1)$, $e_2 \in E(Z_2)$ and $\lambda(e_1) = \lambda(e_2) = 1$. By Definition 3.6 the edges e_1 and e_2 have to cross oddly. Thus, the two cycles have to cross an odd number of times. But on S^2 two cycles, crossing at every intersection in $\mathcal{D}$, have to cross an even number of times. A contradiction. $\square$

We want to introduce two operations, by which we are able to modify drawings. Therefore, we define two switching operations to swap vertices and edges as well as vertices and the crosscap.

Definition 3.11. *Let $\mathcal{D}$ be a drawing of a graph G on S^2 and let v be a vertex and e an edge, that is not incident to v. Then a* vertex-edge switch (v, e) *is pulling a thin finger from the interior of e towards v and let this finger pass over v.*

The execution of a vertex-edge switch (v, e), also known as *finger-move*, changes the crossing parity between e and all edges f incident to v. For all other pairs of edges the crossing parity stays the same, since by the pulling all other crossed edges are crossed pairwise by e, as shown in Figure 3.6. The second operation is the switch of a vertex with the crosscap, where we also alter λ.

Definition 3.12. *Let $(\mathcal{D}, \lambda)$ be a projective HT-drawing of G on S^2. Given a vertex v, the* vertex-crosscap switch *over v is defined as follows: Perform vertex-edge switches (v, e) for all edges e not incident to v with $\lambda(e) = 1$ to obtain a drawing $\mathcal{D}'$ and define a new function $\lambda' : E(G) \to \mathbb{Z}_2$ derived from λ by switching the value of λ on all edges of G incident to v. This yields to a new projective HT-drawing $(\mathcal{D}', \lambda')$.*

Lemma 3.13. *Let $(\mathcal{D}, \lambda)$ be a projective HT-drawing of G on S^2. Let $\mathcal{D}'$ and λ' be obtained from $\mathcal{D}$ and λ by a vertex-crosscap switch. Then $(\mathcal{D}', \lambda')$ is a projective HT-drawing of G on S^2.*

Proof. Let v be the vertex inducing the switch. We will check, that $\mathrm{cr}_{\mathcal{D}'}(e, f) = \lambda'(e)\lambda'(f)$ for any pair of independent edges e and f. First we consider the case, that both e and f are not incident to v, then the crossing-parity of e and f is not changed by the operation, and we get

$$\mathrm{cr}_{\mathcal{D}'}(e, f) = \mathrm{cr}_{\mathcal{D}}(e, f) = \lambda(e)\lambda(f) = \lambda'(e)\lambda'(f).$$

Consider now the other case, that one of the two edges is incident to v. Let e be this edge. Then via the vertex-crosscap switch, we obtain that $\lambda(e) = 1 - \lambda'(e)$ and $\lambda(f) = \lambda'(f)$. Let $\lambda(f) = 0$, then

$$\mathrm{cr}_{\mathcal{D}'}(e, f) = \mathrm{cr}_{\mathcal{D}}(e, f) = \lambda(e)\lambda(f) = 0 = \lambda'(e)\lambda'(f).$$

Otherwise let $\lambda(f) = 1$, then

$$\mathrm{cr}_{\mathcal{D}'}(e, f) = 1 - \mathrm{cr}_{\mathcal{D}}(e, f) = 1 - \lambda(e)\lambda(f) = \lambda(f) - \lambda(e)\lambda(f) = (1 - \lambda(e)) \cdot \lambda(f) = \lambda'(e)\lambda'(f).$$

Hence, all cases are checked and we have shown, that $\mathrm{cr}_{\mathcal{D}'}(e, f) = \lambda'(e)\lambda'(f)$ for any pair of independent edges e and f. □

The vertex-crosscap switch over v does not change the triviality or nontriviality of a cycle Z. If Z contains v, $\lambda(Z) = \lambda'(Z)$, since $\lambda(e) \neq \lambda'(e)$ for exactly two edges e of Z. If otherwise Z does not contain v, $\lambda(Z) = \lambda'(Z)$, since all edges e of Z are not incident to v and hence $\lambda(e) = \lambda'(e)$ for all these edges.

We want to use the vertex-crosscap switch to planarize special subgraphs of the graph G. For this, let $(\mathcal{D}, \lambda)$ be a projective HT-drawing of G on S^2 and let P be a subgraph of G such that every cycle in P is trivial. Then P behaves like a planar subgraph of G, as shown by the following lemma.

Lemma 3.14. *Let $(\mathcal{D}, \lambda)$ be a projective HT-drawing of G on S^2 and let P be a subgraph of G such that every cycle in P is trivial. Then there is a vertex-set $U \subseteq V(P)$ with the following property. Let $(\mathcal{D}_U, \lambda_U)$ be obtained from $(\mathcal{D}, \lambda)$ by vertex-crosscap switches over all vertices of U in any order. Then $(\mathcal{D}_U, \lambda_U)$ is a projective HT-drawing of G on S^2 and $\lambda_U(e) = 0$ for any edge e of $E(P)$.*

Proof. Since $(\mathcal{D}_U, \lambda_U)$ is obtained from $(\mathcal{D}, \lambda)$ by vertex-crosscap switches, $(\mathcal{D}_U, \lambda_U)$ is a projective HT-drawing by Lemma 3.13. Let F be a spanning forest of P, where the spanning trees of P are rooted arbitrarily. THe first step is, that $\lambda(e) = 0$ for every edge e of F. Therefore, we perform a breadth-first search on each tree of h. If we find an edge e with $\lambda(e) = 1$, we perform a vertex-crosscap switch over the vertex of e, which is farther away from the root of the tree. Doing this for all edges of F, we get a map λ_U, where for all edges e of F the value $\lambda_U(e) = 0$ holds. For all other edges f in $E(P) \setminus E(Z)$ there is a cycle Z_f such that $Z_f - f \subseteq F$. But then we get $\lambda_U(f) = 0$ as well, since $\lambda_U(Z_f) = \lambda(Z_f) = 0$. □

3.2.2 Separation Theorem

The idea of the separation theorem is, to split the graph along a simple cycle splits the graph into an inside and outside part. Note that a simple cycle is even and free of self-intersections. Hence, we can work on the two parts separately and merge them afterwards. For this, we define two subgraphs.

Definition 3.15. *Let G be a graph and $\mathcal{D}$ be a drawing of G on S^2. Let Z be a cycle of G such that every edge of Z is even and Z is drawn as a simple cycle in $\mathcal{D}$. Let S^+ and S^- be the two components of $S^2 \setminus \mathcal{D}(Z)$. A vertex $v \in V(G) \setminus V(Z)$ is an* inside vertex *if it belongs to S^+ and an* outside vertex *otherwise. An edge $e = uv \in E(G) \setminus E(Z)$ is an* inside edge *if either u is an inside vertex or if $u \in V(Z)$ and $\mathcal{D}(e)$ points locally to S^+ next to $\mathcal{D}(u)$. Analogously we define an* outside edge*. The sets V^+ and E^+ are the inside vertices and the inside edges, respectively. Analogously, we define V^- and E^- as outside vertices and outside edges. Moreover, we define the two subgraphs $G^{+0} := (V^+ \cup V(Z), E^+ \cup E(Z))$ and $G^{-0} := (V^- \cup V(Z), E^- \cup E(Z))$.*

Note that an edge $e \in E(G) \setminus E(Z)$ is either an inside or an outside edge, since the edges of Z are even. Given this definition, we can formulate the main technical tool for projective HT-drawings.

Theorem 3.16. *Let $(\mathcal{D}, \lambda)$ be a projective HT-Drawing of a 2-connected graph G on S^2 and let Z be a cycle of G that is simple in $\mathcal{D}$ and where every edge is even. Moreover, we assume that every edge e of Z is trivial, that is $\lambda(e) = 0$. Then there is a projective HT-drawing $(\mathcal{D}', \lambda')$ of G on S^2 satisfying the following properties.*

- *The drawings $\mathcal{D}$ and $\mathcal{D}'$ coincide on Z;*
- *the cycle Z is completely free of crossings and all of its edges are trivial in $\mathcal{D}'$;*
- *$\mathcal{D}'(G^{+0})$ is contained in $S^+ \cup \mathcal{D}'(Z)$;*
- *$\mathcal{D}'(G^{-0})$ is contained in $S^- \cup \mathcal{D}'(Z)$; and*
- *either all edges of G^{+0} or all edges of G^{-0} are trivial (according to λ'); that is, at least one of the drawings $\mathcal{D}'(G^{+0})$ or $\mathcal{D}'(G^{-0})$ is an ordinary HT-drawing on S^2.*

The assumption about G that it is 2-connected is not necessary for the proof of Theorem 3.16, but it simplifies some ongoing steps afterwards. To prove Theorem 3.16, we often need that G, $(\mathcal{D}, \lambda)$ and Z fulfill the assumptions of the theorem. Thus, we combine the assumptions in the following definition.

Definition 3.17. *A graph G, a projective HT-drawing $(\mathcal{D}, \lambda)$ and a cycle Z satisfy the* separation assumptions *if*

(1) G is a 2-connected graph;

(2) $(\mathcal{D}, \lambda)$ is a projective HT-drawing of G;

(3) Z is a cycle in G drawn as a simple cycle in $\mathcal{D}$;

(4) every edge of Z is even in $\mathcal{D}$ and trivial

We fix G, $(\mathcal{D}, \lambda)$ and Z satisfying the seperation assumptions, which also fixes an outside and an inside corresponding to the cycle Z.

Definition 3.18. *A* bridge *B of G with respect to Z is a special subgraph of G. Thereby, B is either an edge not in Z but with both endpoints in Z and the endpoints also belong to B, or a connected component of $G - V(Z)$ together with all edges and their endpoints, where one endpoint is in that component and the other one is in Z.*
B is an inside bridge *if it is a subgraph of G^{+0} and an* outside bridge *if it belongs to G^{-0}. A walk ω in G is a* proper walk *if no vertex in ω belongs to $V(Z)$, except possibly its endpoints, and no edge of ω belongs to $E(Z)$. In particular, each proper walk belongs to a single bridge.*

Note that by the assumed 2-connectivity of G, every inside bridge contains at least two vertices of Z. Moreover, the bridges divide the edges and vertices of $G - Z$ into partitions. In the following we want to distinguish, which pairs of vertices on Z are connected by a nontrivial and proper walk via the inside or the outside of Z. For this, we define two, so called, *arrow graphs* A^+ and A^-. The arrow graph A^+ is the *inside arrow graph* for the inside and A^- is the *outside arrow graph* for the outside. These graphs can have loops, but do not have multiple edges. The name results from the idea that the edges are drawn as arrows to distinguish them from $E(G)$. The arrow graphs have the vertex set

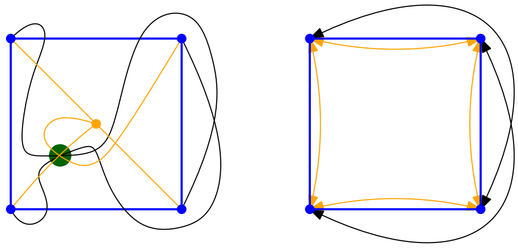


Figure 3.7: A projective HT-drawing of K_5 on the left side with the corresponding inside and outside arrows on the right. The separating cycle Z is shown in blue.

$V(A^+) = V(A^-) = V(Z)$.
The *arrows* correspond directly to the proper nontrivial walks of G. Let u and v be two not necessarily distinct vertices of Z. Let W^+_{uv} be the set of all proper nontrivial walks in G^{+0} with endpoint u and v. Then there is an *inside arrow* $\overline{uv} = \overline{vu}$ in $E(A^+)$ if and only if W^+_{uv} is not empty. Analogously, we define W^-_{uv} and the *outside arrows*. Figure 3.7 shows the arrows corresponding to a projective HT-drawing of a K_5, where the cycle Z is blue, the inside arrows are orange and the outside arrows are black.
Since any walk $w \in W^+_{uv}$ lives in exactly one bridge, we can decompose the set W^+_{uv}. Let B be an inside bridge. Then the set $W^+_{uv,B}$ contains all walks $w \in W^+_{uv}$ such that w belongs to B. Therefore, W^+_{uv} is a disjoint union of the sets $W^+_{uv,B_1}, \ldots, W^+_{uv,B_k}$ for all existing inside bridges $B_1, \ldots, B_k$. An inside arrow $\overline{uv}$ *is induced by* a bridge B if $W^+_{uv,B}$ is not empty. Hence, an inside bridge B is *nontrivial*, if it induces at least one arrow. Two inside arrows $\overline{uv}$ and $\overline{xy}$ are *induced by different bridges*, if there exist at least two different bridges B and B' such that $W^+_{uv,B}$ and $W^+_{xy,B'}$ are not empty. This definition does not exclude, that a bridge induces both $\overline{uv}$ and $\overline{xy}$ even if $\overline{uv}$ and $\overline{xy}$ are induced by different bridges. Analog definitions are made for the outside.

In the following, we introduce forbidden arrow combinations, that means combinations of arrows, which are not possible. On the other hand, there are redrawable arrow combinations, for which we can redraw the subgraph of G without the crosscap.
Again we assume to work with fixed G, $(\mathcal{D}, \lambda)$ and Z satisfying the separation assumption from Definition 3.17. Then following lemma stats properties, which must be fulfilled by the arrow graphs A^+ and A^- of G.

Lemma 3.19. *Let G, $(\mathcal{D}, \lambda)$ and Z be fixed, satisfying the separation assumption. Then the arrow graphs must fulfill the following properties:*

- *Every inside arrow shares a vertex with every outside arrow.*
- *Let $\overline{ab}$ and $\overline{xy}$ be two disjoint inside arrows induced by different inside bridges of G^{+0}. If the two arrows do not share an endpoint, their endpoints have to interleave along Z.*
- *There are no three vertices a, b, c on Z, an inside brigde B^+ and an outside bridge B^- such that $B+$ induces the inside arrows $\overline{ab}$ and $\overline{ac}$ (and no other arrows), and B^- induces the outside arrows $\overline{ab}$ and $\overline{bc}$ (and no other arrows).*

Analogously, the lemma can be formulated with inside and outside swapped.

In Figure 3.8 some forbidden Examples of arrow graphs are shown, which do not fulfill the properties of Lemma 3.19. In (a) not every inside arrow share a vertex with every outside arrow. Part (b) shows an example of inside arrows induced by different bridges. These arrows do not interleave along the cycle Z. And finally the last one sketches the situation

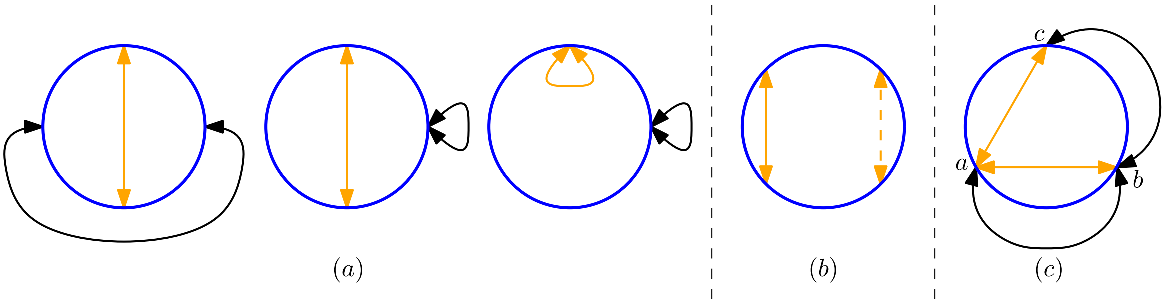


Figure 3.8: Forbidden arrow combinations. In (a) the cyclic order is arbitrary. In (b) the cyclic order is important such that the arrows to not interleave. Different drawn arrows are induced by different bridges. In (c) the inside arrows and the outside arrows are induced by one bridge, respectively.

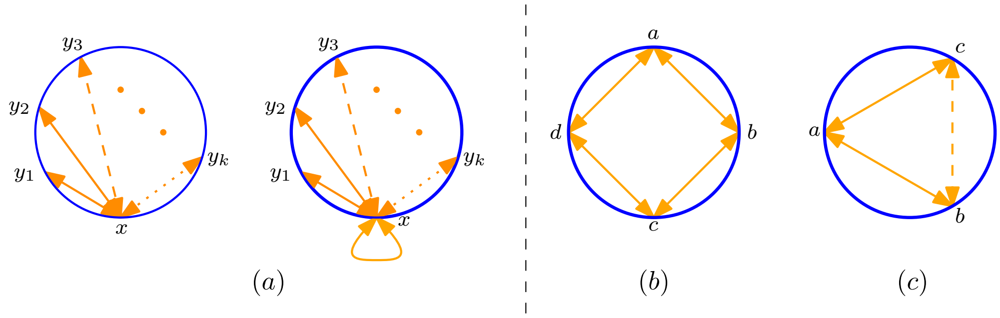


Figure 3.9: The drawings are intended to suggest the redrawable arrow combinations. Different drawn arrows are induced by different bridges. The loop in (a) is an inside arrow drawn outside to keep it recognizable. In (c) only one possible inside split triangle is shown.

of the third point in Lemma 3.19. A proof why these constellations are not possible can be found by using homology and intersection forms [CKP$^+$17, Section 6].
However, there are also arrow combinations, which are redrawable. That means, we can redraw the subgraph of G corresponding to the given arrow graph without using the crosscap. Thus, we can find an ordinary HT-drawing for this subgraph on S^2.

Definition 3.20. *Let G, $(\mathcal{D}, \lambda)$ and Z be fixed, satisfying the separation assumption. Then G forms*

- *an* inside fan, *if all inside arrows shares one common endpoint. (The arrows may come from various inside bridges.)*
- *an* inside square *if G contains four vertices a, b, c, d ordered in this cyclic order along Z and the inside arrows are precisely $\overline{ab}, \overline{bc}, \overline{cd}$ and $\overline{ad}$. In addition, we require that the inside graph G^{+0} contains only one nontrivial inside bridge.*
- *an* inside split triangle *if there exist three vertices a, b, c on Z such that the inside arrows of G are $\overline{ab}, \overline{ac}$ and $\overline{bc}$. In addition, we require that every nontrivial inside bridge induces either the two arrows $\overline{ab}$ and $\overline{ac}$, or just a single arrow.*

Analogously, the outside fan, outside sqaure *and* outside split triangle *are defined.*

In Figure 3.9 two inside fans, an inside square and an inside split triangle are sketched. But the more interesting fact about the definition above is stated by the following two propositions. If there is a projective HT-drawing of G with a fixed cycle Z satisfying the separation assumptions, G contains a constellation from Definition 3.20, which then can be redrawn without using the crosscap.

Proposition 3.21. *Let $(\mathcal{D}, \lambda)$ be a projective HT-drawing on S^2 of a graph G and let Z be a cycle in G satisfying the separation assumptions. Then G forms an inside or outside fan, square or split triangle.*

This proposition can be proved by a relatively direct case analysis using Lemma 3.19 [CKP$^+$17, Section 5]. The second proposition states, that if there is such a constellation from Definition 3.20, we are able to embed the corresponding subgraph of G without using a crosscap.

Proposition 3.22. *Let $(\mathcal{D}, \lambda)$ be a projective HT-drawing of G^{+0} on S^2 and Z be a cycle satisfying the separation assumptions. Let $\mathcal{D}(G^{+0}) \cap S^- = \emptyset$ (that is, G^{+0} is fully drawn on $S^+ \cup \mathcal{D}(Z)$). Moreover, let G^{+0} form an inside fan, inside square or inside split triangle. Then there is an ordinary HT-drawing $\mathcal{D}'$ of G^{+0} on S^2 such that $\mathcal{D}$ coincides with $\mathcal{D}'$ on Z and $\mathcal{D}'(G^{+0}) \cap S^- = \emptyset$.*

Again, Proposition 3.22 is proven by treating each constellation separately [CKP$^+$17, Section 7]. Thereby, for each one is exactly shown, how to redraw the corresponding graph G^{+0} without using the crosscap, such that we get an ordinary HT-drawing for G^{+0}.
Finally, the last thing we need is the “redrawing procedure” of Pelsmajer, Schaefer and Štefankovič [PSŠ07a, Theorem 2.1]. The theorem is not stated in the setting of projective HT-drawings, which is fine due to Definition 3.15.

Theorem 3.23. *Let $\mathcal{D}$ be a drawing of a graph G on the sphere S^2. Let Z be a cycle in G such that every edge of Z is even and Z is drawn as a simple cycle.*
Then there is a drawing $\mathcal{D}''$ of G such that

- *$\mathcal{D}''$ coincides with $\mathcal{D}$ on Z;*
- *$\mathcal{D}''(G^{+0})$ belongs to $S^+ \cup \mathcal{D}(Z)$ and $\mathcal{D}''(G^{-0})$ belongs to $S^- \cup \mathcal{D}(Z)$;*
- *whenever (e, f) is a pair of edges such that both e and f are inside edges or both are outside edges, then $\mathrm{cr}_{\mathcal{D}''}(e, f) = \mathrm{cr}_{\mathcal{D}}(e, f)$.*

The idea of the proof of Theorem 3.23 is to move all crossings between $E(G - Z)$ and $E(Z)$ to one edge of Z. And afterwards, map G^{+0} to S^+ and G^{-0} to S^- via a self-homomorphism, such that the number of crossings between pairs of edges on each side is unchanged [CKP$^+$17, Section 8].
Given these statements, we can prove the separation theorem.

Theorem 3.16. *Let $(\mathcal{D}, \lambda)$ be a projective HT-Drawing of a 2-connected graph G on S^2 and let Z be a cycle of G that is simple in $\mathcal{D}$ and where every edge is even. Moreover, we assume that every edge e of Z is trivial, that is $\lambda(e) = 0$. Then there is a projective HT-drawing $(\mathcal{D}', \lambda')$ of G on S^2 satisfying the following properties.*

- *The drawings $\mathcal{D}$ and $\mathcal{D}'$ coincide on Z;*
- *the cycle Z is completely free of crossings and all of its edges are trivial in $\mathcal{D}'$;*
- *$\mathcal{D}'(G^{+0})$ is contained in $S^+ \cup \mathcal{D}'(Z)$;*
- *$\mathcal{D}'(G^{-0})$ is contained in $S^- \cup \mathcal{D}'(Z)$; and*

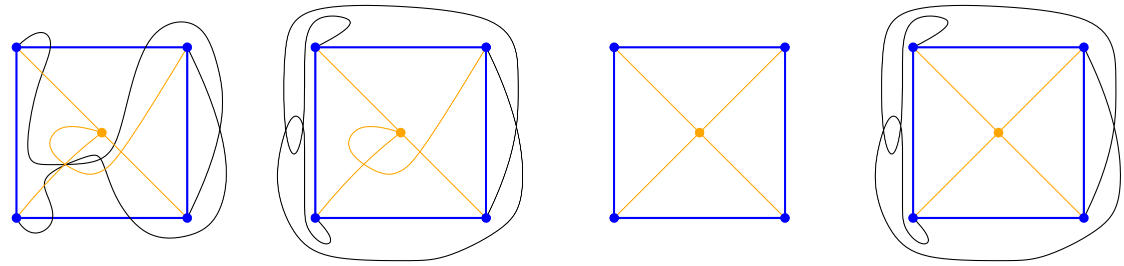


Figure 3.10: Redrawing steps on the example of a K_5 according to the proof of Theorem 3.16.

- *either all edges of G^{+0} or all edges of G^{-0} are trivial (according to λ'); that is, at least one of the drawings $\mathcal{D}'(G^{+0})$ or $\mathcal{D}'(G^{-0})$ is an ordinary HT-drawing on S^2.*

Proof of Theorem 3.16. Let G be a graph and $(\mathcal{D}, \lambda)$ be a projective HT-drawing of G. Let Z be a cycle of G, such that G, $(\mathcal{D}, \lambda)$ and Z fulfill the separation assumptions from Definition 3.17. Since Z is even and drawn as a simple cycle, we can use Theorem 3.23 to get a drawing $\mathcal{D}''$ of G, as shown in Figure 3.10. Afterwards, Z is crossing free and separates the graph into G^{+0} and G^{-0}. Thereby, $\mathcal{D}''(G^{+0})$ is contained in $S^+ \cup \mathcal{D}(Z)$ and $\mathcal{D}''(G^{-0})$ in $S^- \cup \mathcal{D}(Z)$. To get a projective HT-drawing $(\mathcal{D}'', \lambda'')$ we need a map λ'', but there may be no such map. Thus, we need to modify $\mathcal{D}''$.
Applying Proposition 3.21 to $(\mathcal{D}, \lambda)$ stats, that on one side of Z the graph G induces a fan, square or split triangle. Let the redrawable combination be without loss of generality on the inside. Therefore, $\mathcal{D}''(G^{+0})$ satisfies the assumptions of Proposition 3.22. This results in an ordinary HT-drawing $\mathcal{D}^+(G^{+0})$ for G^{+0} such that $\mathcal{D}''(Z) = \mathcal{D}^+(Z)$. Given the drawings $\mathcal{D}''$ and $\mathcal{D}^+$ we combine them at Z to create $\mathcal{D}'$. Then $\mathcal{D}'$ is a drawing of G on S^2, where G^{+0} is drawn as in $\mathcal{D}^+$ and G^{-0} as in $\mathcal{D}''$. The matching is well defined, since both drawings coincide on Z as already described.
To get a projective HT-drawing, we still need a map λ'. We set λ' to be $\lambda'(e) := \lambda(e)$ for $e \in E^-$ and $\lambda'(e) := 0$ otherwise. To show that λ' is indeed the expected map, let e and f be to independent edges of G. Let both be inside edges. Since $\mathcal{D}^+$ is an ordinary HT-drawing, both edges do not cross. So, we get

$$\mathrm{cr}_{\mathcal{D}'}(e, f) = \mathrm{cr}_{\mathcal{D}^+}(e, f) = 0 = \lambda'(e)\lambda'(f).$$

Assume otherwise, that e and f are both outside edges, then

$$\mathrm{cr}_{\mathcal{D}'}(e, f) = \mathrm{cr}_{\mathcal{D}''}(e, f) = \lambda(e)\lambda(f) = \lambda'(e)\lambda'(f).$$

If now one edge is an inside edge and one is an outside edge, these edges do not cross since $\mathcal{D}'(e)$ and $\mathcal{D}'(f)$ are separated by $\mathcal{D}'(Z)$. So we get

$$\mathrm{cr}_{\mathcal{D}'}(e, f) = 0 = \lambda'(e)\lambda'(f).$$

□

3.2.3 Proof of the Strong Hanani-Tutte Theorem on $\mathbb{R}P^2$

Now we finally prove Theorem 3.4, given the previous results. For this, we show that a graph G with an HT-drawing on $\mathbb{R}P^2$ is actually projective-planar. We can assume that G has a projective HT-drawing $(\mathcal{D}, \lambda)$ on S^2 by Corollary 3.8. The goal is to use Theorem 3.16, but for this G must fulfill the separation assumptions from Definition 3.17. We start with a decomposition of G into blocks of 2-connectivity.

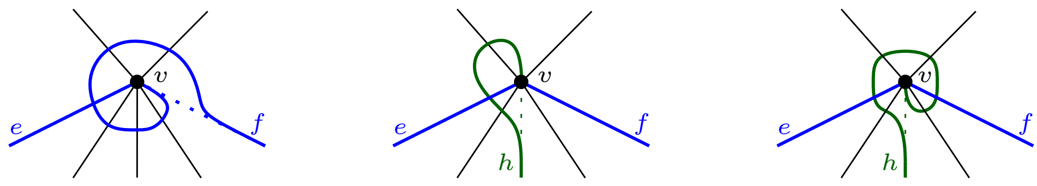


Figure 3.11: Local redrawing operations at Z (blue). The dotted edges represent the original course of the edges.

Lemma 3.24. *If G admits a projective HT-drawing on S^2, then at most one block of G is non-planar. Moreover, if all blocks are planar, G is planar as well.*

Proof. Let G be a graph with a projective HT-drawing $(\mathcal{D}, \lambda)$ on S^2. Let B_1 and B_2 be two distinct non-planar blocks of G. Assume that B_1 and B_2 do not share a vertex v. Then at least one of the two blocks does not contain any non-trivial cycle by Lemma 3.10. Without loss of generality, let B_2 be this block. But then B_2 is a subgraph of G where every cycle is trivial. Then by Lemma 3.14, the block B_2 can be drawn on S^2 without crossings, such that its drawing $\mathcal{D}(B_2)$ is an ordinary HT-Drawing. Thus, we can use the strong Hanani-Tutte theorem for the plane (Theorem 1.2) to prove that B_2 is planar. This is a contradiction to the assumption that both B_1 and B_2 are non-planar.
Assume otherwise that B_1 and B_2 share a cut vertex v. We consider the graph $H := B_1 \cup B_2$ together with its spanning tree P. Let P be chosen, such that v is only incident to two edges $e_1 \in B_1$ and $e_2 \in B_2$. This is possible, since v is a cut-vertex and B_1 and B_2 are not connected after removing v. By using Lemma 3.14 again, we can make all edges of P trivial by possibly alternating λ. Then any non-trivial edge e in $E(H) \setminus E(P)$ forms a non-trivial cycle Z_e in one of the blocks. If the picked edge e is not incident to v, then Z_e does not contain v as we have selected P. Hence, by Lemma 3.10 in at least one block all non-trivial edges are incident to v. Let this block again be B_2. Then B_2 is already planar, since by $\mathcal{D}$ an HT-drawing of B_2 on S^2 is already given where we have no pairs of non-trivial independent edges. This contradicts the assumption, that B_1 and B_2 are non-planar.
The second statement of the lemma follows directly by Lemma 2.2. □

Lemma 3.25. *Let $(\mathcal{D}, \lambda)$ be a drawing of a 2-connected graph. If $\mathcal{D}$ does not contain any trivial cycle, then G is planar.*

Proof. Let G be a 2-connected graph. Then G is either a cycle or it contains two vertices connected by three disjoint paths. A cycle is already planar. For the other case, at least two of the paths are trivial or at least two are non-trivial by the pigeonhole principle. But then these two paths form a trivial cycle, which cannot occur due to the assumption. □

Given the needed results about 2-connectivity, we continue with redrawing a cycle Z such that it fulfills the separation assumptions.

Lemma 3.26. *Let $(\mathcal{D}, \lambda)$ be a projective HT-drawing of a graph G on S^2 and let Z be a cycle in G. Then G can be redrawn only by local changes next to the vertices of Z to a projective HT-drawing $(\mathcal{D}', \lambda)$ on S^2 so that λ remains unchanged and $\mathrm{cr}_{\mathcal{D}'}(e, f) = \lambda(e)\lambda(f)$ for any pair $(e, f) \in E(Z) \times E(G)$ of distinct (not necessarily independent) edges. In particular, if $\lambda(e) = 0$ for every edge e of Z, then every edge of Z is even in $\mathcal{D}'$.*

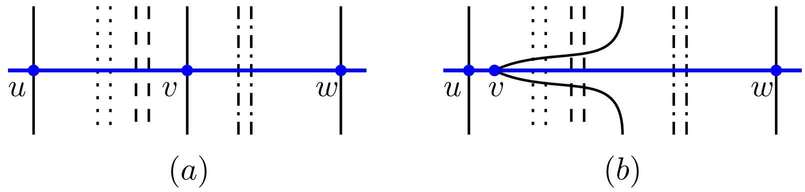


Figure 3.12: Moving crossings along Z such that only one edge of Z is intersected by other edges.

Proof. The goal is to show, that we can apply local changes to $\mathcal{D}$, such that the crossing parity of $\mathrm{cr}_{\mathcal{D}}(e, f)$ changes, if e is an edge of Z and e and f are adjacent. To achieve that $\mathrm{cr}_{\mathcal{D}'}(e, f) = \lambda(e)\lambda(f)$ for any pair $(e, f) \in E(Z) \times E(G)$ of distinct not necessarily independent edges, we have to change the crossing parity between the edges of Z and their adjacent edges.
At first, we want to fulfill the equation for consecutive edges of Z. Thus, let e and f be consecutive edges on Z with common vertex v. If $\mathrm{cr}_{\mathcal{D}}(e, f) \neq \lambda(e)\lambda(f)$, we locally redraw f around v, such that we add a crossing between e and f, as shown in Figure 3.11 (a). Thereby the crossing parity between edges on Z and dependent edges not on Z can change. Now we fix these crossing parities that may have changed to the wrong value during the procedure. We consider three dependent edges e, f and h, where e and f are on Z and h is not on Z. Let v be their common vertex. If we now have to change either the parity of $\mathrm{cr}_{\mathcal{D}}(e, h)$ or the parity of $\mathrm{cr}_{\mathcal{D}}(f, h)$ we redraw h around v to change the crossing parity between h and either e or f. The operation is shown in Figure 3.11 (b) for the redrawing with e. For h and f a symmetric operation is used. If the crossing parity of both pairs (e, h) and (f, h) have to be changed, we redraw h once completely around v as shown in Figure 3.11 (c). These moves do not change the parity of $\mathrm{cr}_{\mathcal{D}}(e, h')$ or $\mathrm{cr}_{\mathcal{D}}(f, h')$ for any other edge h'. Hence, step by step the crossing parities change such that $\mathrm{cr}_{\mathcal{D}'}(e, f) = \lambda(e)\lambda(f)$ is fulfilled for any pair $(e, f) \in E(Z) \times E(G)$. □

Given this first step to make the edges of a cycle made even, the next step is to make this cycle simple. Note that a cycle is simple if all its edges are even and it is drawn without self-intersections.

Lemma 3.27. *Let $(\mathcal{D}, \lambda)$ be a projective HT-drawing on S^2 of a graph G and let Z be a cycle in G such that each of its edges is even. Then G can be redrawn so that Z becomes a simple cycle, its edges remain even and the resulting drawing is still a projective HT-drawing (with λ unchanged).*

Proof. To prove this lemma we firstly remodel $\mathcal{D}$, such that Z is free of crossings, except for one edge. Consider three consecutive vertices u, v and w on Z. By *almost-contracting* the edge uv, we make uv crossing free. For this, we move v and its incident edges along uv towards u until all intersections between uv and other edges are shifted to vw, as shown in Figure 3.12. This operation does not change the image of Z. Since both edges uv and vw were even edges in $\mathcal{D}$, vw is still even after the redrawing step. By the same argumentation the crossing parity between edges incident to v and other edges is not affected. If uv and vw intersect each other, we introduce a self-intersection of vw. By repeatedly doing such redrawings, we achieve a drawing where only one edge of Z is intersected.
The (possible inserted) self-intersections of Z can be removed as described in Chapter 2 and shown in Figure 2.2. Hence, we get a drawing $\mathcal{D}'$, where Z is drawn as a simple cycle, and since we did not change the crossing parities, $(\mathcal{D}', \lambda)$ is still a projective HT-drawing. □

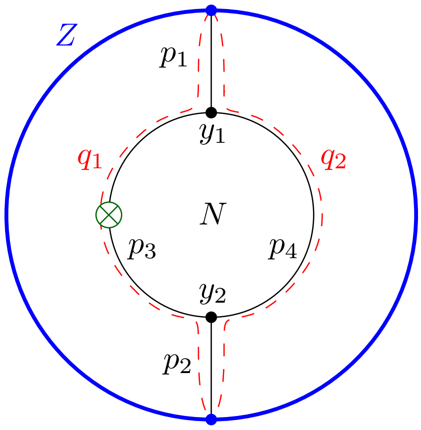


Figure 3.13: Sketch of the setting, that B does not contain a non-trivial cycle.

After introducing these lemmata to obtain the separation assumptions, let G, $(\mathcal{D}, \lambda)$ and Z fulfill the separation assumptions of Definition 3.17. The following lemma will be helpful in the proof of Theorem 3.4, it connects a bridge to a single arrow in the arrow graph. Note that the arrow graph of G contains the vertices of Z and the edges are represented by arrows.

Lemma 3.28. *Let G, $(\mathcal{D}, \lambda)$ and Z satisfy the separation assumptions. Let B be an inside bridge such that any proper path in B with both endpoints on $V(B) \cap V(Z)$ is non-trivial. Then $|V(B) \cap V(Z)| = 2$ and B induces a single arrow and no loop.*

Proof. We first show that the given bridge B does not contain a non-trivial cycle. For contradiction, we assume B contains a non-trivial cycle N. Since G is 2-connected, there are two vertex-disjoint paths p_1 and p_2 connecting Z and N. We assume that these two paths are minimal connecting N and Z. That means, both paths p_1 and p_2 share exactly one vertex with both Z and N. Let y_1 and y_2 be the vertices that N share with p_1 and p_2, respectively. The vertices y_1 and y_2 divide N into two subarcs, which we call p_3 and p_4. Without loss of generality, let p_3 be the arc that crosses the crosscap an odd number of times. Given this setting, we consider two paths q_1 and q_2. Thereby, $q1$ is a concatenation of p_1, p_3 and p_2, and q_2 is a concatenation of p_1, p_4 and p_2. The setting is sketched in Figure 3.13. But then q_2 is a trivial path. This contradicts the assumption of the lemma, that any proper walk in B is non-trivial. Hence, B does not contain any non-trivial cycle.
Secondly, we show that B does not induce a loop in the inside arrow graph. We prove this claim again by contradiction. Assume B induces an arrow loop at a vertex x on Z. Hence, there exists a proper non-trivial walk κ in B. Let κ be the shortest such walk. By the previous paragraph, κ can not be a cycle. Thus, it must contain a closed nonempty subwalk κ'. Let κ' again be the shortest such walk. So, κ' must be a cycle, which is trivial by the previous paragraph. But then we can shorten κ by κ' and get a contradiction to the minimality of κ.
It remains to prove, that $|V(B) \cap V(Z)| = 2$. Since G is 2-connected, we know that $|V(B) \cap V(Z)| \geq 2$ holds. To prove that $|V(B) \cap V(Z)| \leq 2$, assume that a, b and c are three distinct vertices of $V(B) \cap V(Z)$. Since in this case B is not a single edge, there exists an inner vertex v of B. By Definition 3.18 a bridge contains proper walks p_a, p_b and p_c connecting v with the corresponding vertex on Z. Each of these walks is either trivial or non-trivial. By the pigeonhole principle, at least two of the walks have either 0 or 1 as the value of λ. Let p_a and p_b have the same value. But then the concatenation of p_a and p_b creates a trivial proper walk in B, that can be shortened to a trivial proper path between

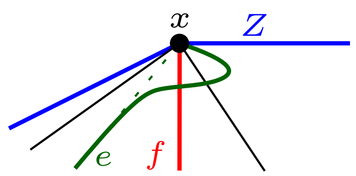


Figure 3.14: Local changes at u, a vertex Z and γ share. The redrawn edge e is printed in green with the original drawing of e dotted.

a and b by the same argumentation as above. Thus, we get again a contradiction. Hence, $|V(B) \cap V(Z)| = 2$.
Let x and y be these two vertices on Z. As proven above, every path connecting x and y is non-trivial. So, B induces a single arrow $\overline{xy}$ in the inside arrow graph A^+. Thereby, no other arrow in $A+$ can be induced by B, since no arrow loop is possible. □

The following proposition is the main tool to prove Theorem 3.4 by using Theorem 3.16. It is proven inductively with the help of Theorem 3.16. We will sketch the proof of the proposition below.

Proposition 3.29. *Let $(\mathcal{D}, \lambda)$ be a projective HT-drawing of a 2-connected graph G on S^2 and Z a cycle in G that is completely free of crossings in $\mathcal{D}$ and such that each of its edges is trivial in $\mathcal{D}$. Assume that (V^+, E^+) or (V^-, E^-) is empty (recall the notation from the definition of inside/outside arrow graph). Then G can be embedded into $\mathbb{R}P^2$ so that Z bounds a face of the resulting embedding that is homeomorphic to a disk. If, in addition, $\mathcal{D}$ is an ordinary HT-drawing on S^2, then G can be embedded into S^2 so that Z bounds a face of the resulting embedding (this face is again homeomorphic to a disk - there is in fact no other option on S^2).*

Sketch of the proof. The proof works by induction over the number of edges of G. The base case is, if G is just a cycle.
We assume without loss of generality, that (V^-, E^-) is empty. Thus, $G = G^{+0}$ and if (V^+, E^+) is also empty, $G = Z$ and we are done. So, we assume (V^+, E^+) is not empty. We pick a path γ in $V(G^{+0}), E(G^{+0} \setminus E(Z))$ connecting two points x and y with $x \neq y$ on Z. Now we consider two cases based on whether there exists a trivial γ or not.

Case 1: Trivial γ exists. First, we want to make γ even and simple, while keeping these properties for Z. Therefore, we use Lemma 3.14 on the inner vertices of γ such that $\lambda(e) = 0$ for all edges e of Z and γ. Then, to make the edge of γ even without making the edges of Z odd, we proceed as in Lemma 3.26. The only place, we have to act differently is for the edges $e \notin E(\gamma) \cup E(Z)$ incident to x or y. We will use without loss of generality the vertex x in the following. Let f be the edge of γ incident to x and e and f cross oddly. Then we use the redrawing move sketched in Figure 3.14. So, we get a drawing where all edges of Z and γ are even. To make γ simple, we use the redrawing procedure from Lemma 3.27 as shown in Figure 3.12. Hence, we get a drawing of G^{+0} where γ is even and simple.
Now we want to switch the inside and outside to simplify the argumentation. This is allowed due to the use of a homeomorphism of S^2. Afterwards, the inside region bounded by Z is empty.
Let x and y lie antipodal on Z. These vertices split Z into two arcs p_1 and p_2, whereby we name p_1 the ‘upper’ one and p_2 the ‘lower’ one. Moreover, we assume that γ lies above p_1. This can be done by the right choice of correspondence between S^2 and the plane. The setting is sketched in Figure 3.15 (a). Now we let the cycle Z collapse, such that p_1 and p_2

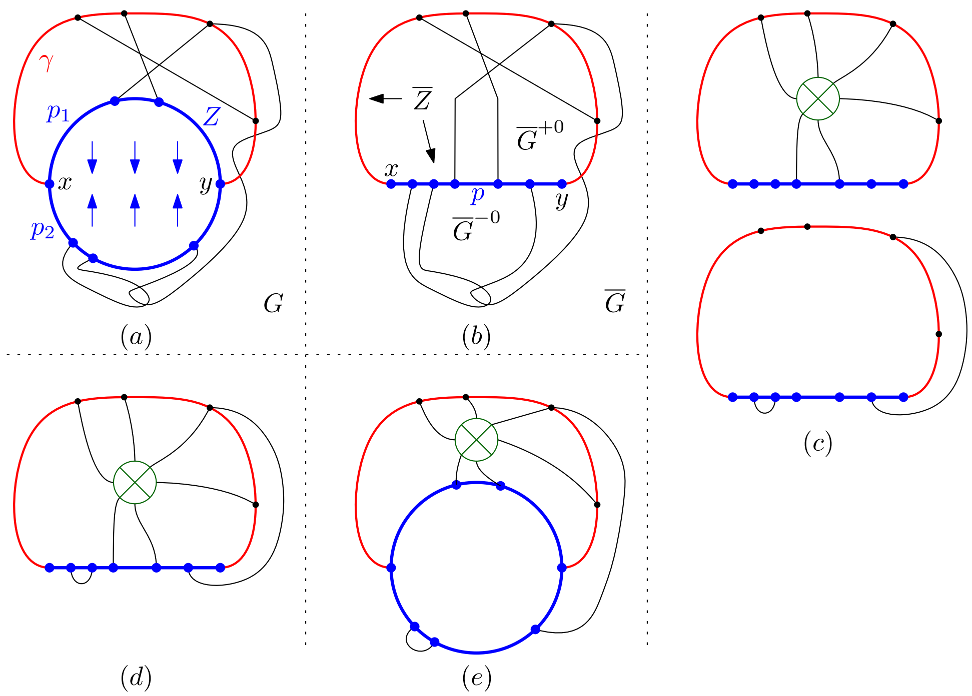


Figure 3.15: The deformation steps of G into $\overline{G}$, splitting the new graph, to get embeddings and put the embeddings back together to achieve an embedding of G.

deform towards the straight connection between x and y. The inner vertices of p_1 and p_2 are also shifted, but in such a way, that they do not lie on the same point. Let us call this resulted connection path p, see Figure 3.15 (a) and (b). This results in a new projective HT-drawing $(\overline{\mathcal{D}}, \overline{\lambda})$ of the new graph $\overline{G}$, where all vertices are preserved. The edges have reduced by one, since the edges of Z have been removed but replaced by the edges of p. For $\overline{\lambda}$ we keep the value for all adopted edges and set $\overline{\lambda}(e) = 0$ if e is an edge of p.
In this new graph $\overline{G}$ the paths γ and p form a cycle $\overline{Z}$. Then $\overline{G}, \overline{Z}$ and $(\overline{\mathcal{D}}, \overline{\lambda})$ fulfill the separation assumptions of Definition 3.17. So, the region between γ and p_1 from G corresponds to $\overline{G}^{+0}$. Thus, we can apply Theorem 3.16 and obtain a drawing $\overline{\mathcal{D}}'$ of $\overline{G}$. Thereby, if we look at the two sides separately, one side, say $\overline{G}^{+0}$ has a projective HT-drawing and the other one, say $\overline{G}^{-0}$, an ordinary HT-drawing on S^2. If D were already an ordinary HT-drawing, both sides result in an ordinary HT-drawing by Theorem 3.23.
Since G is 2-connected, $\overline{G}$ is 2-connected as well. We apply the induction hypothesis to $\overline{G}^{+0}$ and $\overline{G}^{-0}$ separately. Hence, we get an embedding of $\overline{G}^{+0}$ on $\mathbb{R}P^2$ and an embedding of $\overline{G}^{-0}$ on S^2, as shown in Figure 3.15 (c). In both embeddings $\overline{Z}$ bounds a face homeomorphic to a disk. Again, if $\mathcal{D}$ were an ordinary HT-drawing, both embeddings lie on S^2.
Now we merge the two embeddings of $\overline{G}^{+0}$ and $\overline{G}^{-0}$ along $\overline{Z}$ to get an embedding of $\overline{G}$, see Figure 3.15 (d).
To get an embedding of G, we only have to reshape Z. For this, we undo the identification of p_1 and p_2 to p. A vertex v on p belongs to p_1 if all incident edges $e \in E(G) \setminus E(Z)$ belong to $\overline{G}^{+0}$. Analogously, a vertex v on p belongs to p_2 if all incident edges $e \in E(G) \setminus E(Z)$ belong to $\overline{G}^{-0}$. So, we can move the vertices of p with its incident edges to p_1 and p_2 to obtain the original cycle Z. Hence, we have undone the identification and we get an embedding of G for this case, as shown in Figure 3.15 (e).

Case 2: No trivial γ exists. Since each possible path γ is non-trivial, we fulfill the requirements of Lemma 3.28. Thus, every inside bridge B induces exactly one arrow in the inside arrow graph A^+.
We claim, that we can handle each bridge separately. Namely, for any inside bridge B,

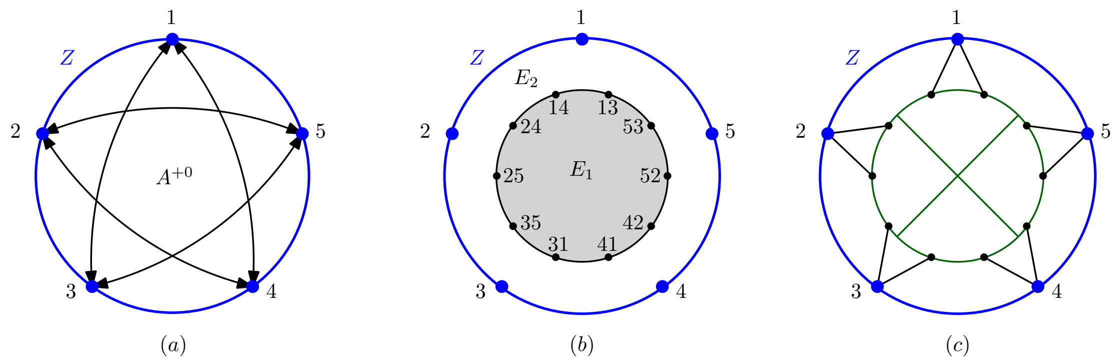


Figure 3.16: An example of a mixed graph A^{+0} together with the redrawing steps.

there is a planar drawing of $Z \cup B$ such that Z is the outer face. This claim holds, since a single arrow forms an inside fan and by Proposition 3.22 this inside fan is redrawable, such that we receive an ordinary HT-drawing. But then we work with an ordinary HT-drawing, which is already dealt with by Case 1.

Let A^{+0} be a mixed graph obtained from A^{+} by adding the edges of Z. So, A^{+0} contains Z and the inside arrows of A^{+}, see Figure 3.16 (a) for an example. The goal is to find an embedding of A^{+0} in $\mathbb{R}P^2$ with Z bounding a face. Afterwards, by the previous claim, we are able to replace the inside arrows by an embedding of the corresponding bridges. If several bridges induce the same arrow, we embed these brides in parallel.

To find an embedding of A^{+0}, we abuse point two of Lemma 3.19 that disjoint inside arrows must interleave. We want to reorder the connections of the arrows to the vertices of Z. Therefore, let E_1 and E_2 be two concentric disks such that E_1 is in the interior of E_2 and Z lies on the boundary of E_2. Let a be the number of inside arrow of A^{+}. Then we put $2a$ points on the boundary of E_1 evenly distributed. We name the points by pairs (x, y) where $\overline{xy}$ is an inside arrow. The names are assigned to the points as follows: The first coordinate is ordered in the same way, as the vertices of Z are ordered along Z. The second coordinate for a fixed first coordinate x is ordered in reverse order corresponding to the order of the vertices of Z along Z, as shown in Figure 3.16 (b). Then the two points (x, y) and (y, x) lie directly opposite on E_1. Then we add simultaneously for each arrow $\overline{xy}$ an edge between x and (x, y) as well as an edge between y and (y, x) without crossings. By identifying the opposite points of E_1 we create a crosscap and get an embedding $\mathcal{E}(A^{+0})$ of A^{+0} on $\mathbb{R}P^2$. By replacing the edges representing arrows in $\mathcal{E}(A^{+0})$ with the embeddings of the corresponding bridges, we are done. $\square$

Given all these results, we can finally prove the strong Hanani-Tutte theorem for the projective plane.

Theorem 3.4. *A graph G can be embedded into the projective plane if and only if it admits a Hanani-Tutte drawing on the projective plane.*

Proof of Theorem 3.4. The proof is done by induction over the number of vertices of G. Assume that G has at least three vertices, since otherwise the theorem follows directly by the simplicity of G. Let us consider two cases based on whether G is 2-connected or G has at least two 2-connected blocks.

We start with the latter one and assume G can be represented as $G_1 \cup G_2$ with a minimal cut $G_1 \cap G_2$ of G. This separation contains at most one cutvertex. Using Lemma 3.24 we can assume that, without loss of generality G_1 is planar and G_2 is non-planar. Then, we get an embedding $\mathcal{E}_2(G_2)$ on $\mathbb{R}P^2$ by induction. Hence, we have an embedding of G_1, an

embedding of G_2 and at most one cutvertex. So, by merging the drawings at the cutvertex we get an embedding of $G = G_1 \cup G_2$ on $\mathbb{R}P^2$.
Let us assume otherwise, that G is 2-connected. Then $(\mathcal{D}, \lambda)$ contains at least one trivial cycle Z by Lemma 3.25. Each edge of Z can be made trivial by Lemma 3.14 and even by Lemma 3.26. Given these properties for the edges of Z, we make Z simple by Lemma 3.27. This yields a projective HT-drawing $(\mathcal{D}^*, \lambda^*)$, which fulfills together with G, and Z the separation assumptions from Definition 3.17. Hence, we can apply Theorem 3.16 to obtain a projective HT-drawing $(\mathcal{D}', \lambda')$ that separates G along Z into G^{+0} and G^{-0}. Let $\mathcal{D}^+ := \mathcal{D}'(G^{+0})$ and $\mathcal{D}^- := \mathcal{D}'(G^{-0})$ such that without loss of generality $\mathcal{D}^-$ is an ordinary HT-drawing on S^2 and D^+ a projective HT-drawing on S^2. By applying Proposition 3.29 to both drawings $\mathcal{D}^-$ and $\mathcal{D}^+$ separately, we get embeddings $\mathcal{E}(G^{+0})$ and $\mathcal{E}(G^{-0})$ of G^{+0} and G^{-0}. Thereby, $\mathcal{E}(G^{+0})$ is on $\mathbb{R}P^2$ and $\mathcal{E}(G^{-0})$ is on S^2. In both embeddings Z bounds a face. Hence, we can merge the two embeddings along Z into one embedding of G on $\mathbb{R}P^2$. □

3.3 Uniform Hanani-Tutte on the Projective Plane

The goal of this section is to deal with the uniform Hanani-Tutte conjecture for the projective plane. The conjecture is the following:

Conjecture 3.30. *Let G be a graph with an independently even drawing on the projective plane. Then G can be embedded on the projective plane, i.e. drawn crossing free, such that the embedding scheme at even vertices is preserved.*

By using the contraction operation described in Section 3.1, we are able to prove a property of a minimal counter example. We modify the given graph G such that even vertices form an independent set. An *even vertex* is thereby a vertex, such that all its incident edges are even. This reduces the problem to a smaller instance. Therefore, we can assume for a minimal counter example that even vertices are not connected by an edge.

Lemma 3.31. *In a minimal counter example the even vertices form an independent set.*

Proof. Let G be a graph and $\mathcal{D}(G)$ a minimal counterexample. Assume for contradiction, that there are two even vertices x and y connected by an even edge e. By contracting e as defined in Section 3.1 and shown in Figure 3.1 we get a new graph G' with a new drawing $\mathcal{D}'(G')$ that contains a new vertex x' with the combined rotations of x and y instead of x, y and e. Since all edges incident to x and y were even, the edges incident to x' stay even. By induction, we get an embedding scheme $\mathcal{E}'(G')$ of G' such that the rotation system at even vertices is preserved. By Lemma 3.1, we can then undo the contraction and get an embedding $\mathcal{E}(G)$ of G preserving the original embedding scheme at even vertices. So, $\mathcal{D}(G)$ is not a minimal counterexample and we have a contraction. Hence, even vertices must form an independent set in $\mathcal{D}(G)$. □

Given this approach of a minimal counter example, we tried to adapt the proof of the strong Hanani-Tutte theorem to obtain a proof for Conjecture 3.30. The huge problem why we cannot use the proof of the strong Hanani-Tutte theorem as a basis for the proof of the uniform version, is the used representation of drawings. The representation as a projective HT-drawing on S^2 possibly adds odd crossings between dependent edges at a vertex v as shown in Figure 3.17. Then we would be allowed to reorder the edges at v. Hence, we cannot ensure that at v the rotation is preserved and the uniform Hanani-Tutte theorem on the projective plane is not fulfilled.

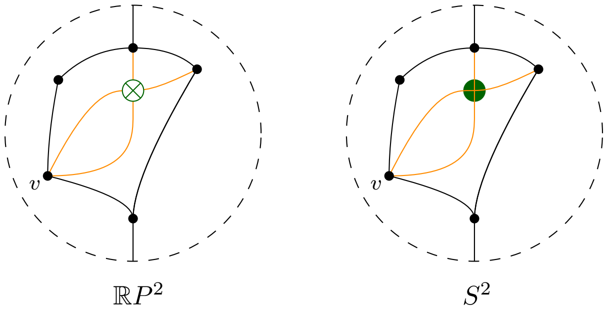


Figure 3.17: Removing the crosscap adds an odd crossing between dependent edges.

Moreover, at several points in the proof, the rotation system is not preserved at even vertices. For example, the redrawable combinations of arrows from Definition 3.20 are not redrawable in the same way as it is proven for the strong version. There we often have to change the rotation at the vertices of Z, where multiple bridges are incident.
So, perhaps an approach to prove Conjecture 3.30 would be to use the forbidden minors for the projective plane as it was done by Pelsmajer, Schaeffer and Stasi for the strong version [PSS09], but there it could be difficult to keep the rotations at even vertices. On the other hand, it is possible, that one has to find a completely new proof independent of the given proofs for the strong version.

In the next Chapter we will treat the Hanani-Tutte theorems for radial planarity. For this, we change the surface we are working on back to the plane. Hence, the setting is easier, since the surface does not have a crosscap.

4. Uniform Hanani-Tutte for Radial Drawings

In this chapter we want to discuss the Hanani-Tutte theorem for radial planarity. Thereby, we start with some definitions, followed by the weak Hanani-Tutte theorem for radial planarity. The presented proof in Section 4.2 is the proof of Fulek, Pelsmajer and Schaefer [FPS17, FPS23]. Afterwards, we show the strong Hanani-Tutte theorem for radial planarity in Section 4.3. This proof is based on the proof given again by Fulek, Pelsmajer and Schaefer [FPS23]. The last section of this chapter will state the uniform Hanani-Tutte theorem.

4.1 Terminology

A given graph G is a *leveled graph*, if its vertices can be assigned to levels $1, \dots, k$, and the edges of G connect vertices on different levels and are monotone. An example of a leveled graph is shown in Figure 4.1 (a). The same graph G can be represented as a *radial drawing*. Therefore, the levels are drawn as concentric circles, meaning they share the same centre. The edges of G in the radial drawing have to be *radial*, so they intersect each circle at most once. Figure 4.1 (b) shows a radial representation. If there is a one on one matching between the vertices and the levels (each vertex has its one level), we say, that G is an *ordered graph*. In such an ordered graph G there is a lowest and highest vertex, namely v_1 and v_n. Analogously to Chapter 2, a drawing D is an *(independently) even radial drawing*, if it is a radial drawing, such that all (independent) edges cross an even number of times. A graph G is *radial-planar*, if there exists a radial embedding $\mathcal{E}(G)$ of G. That is a crossing-free drawing of G respecting the levels. It is clear, that every level-planar graph is also radial-planar, since we can place the planar drawing of a level-planar graph also on the concentric cycles without changing the embedding. But the other way around does not hold, as Figure 4.2 shows.
For this chapter, we allow that our considered graphs have multiple edges, but no loops. For the sake of simplicity, we use a special representation of radial drawings. We draw the graphs on a standing *cylinder* $\mathcal{C} = \mathbb{S}^1 \times (0,1) = \{(\cos\Theta, \sin\Theta, \ell) : \Theta \in \mathbb{R}, \ell \in (0,1)\}$. A vertex on the cylinder is specified by (Θ, ℓ). An edge is radial, if there is a function f with interval domain such that $\Theta = f(\ell)$ is fulfilled. A drawing is radial on the cylinder, if the ℓ-coordinates of the vertices match the levels and the edges are radial. For a better visibility of the edges we use the flat version of the cylindrical representation for figures as shown in Figure 4.3. There we cut the cylinder open and roll it out such that we get a flat

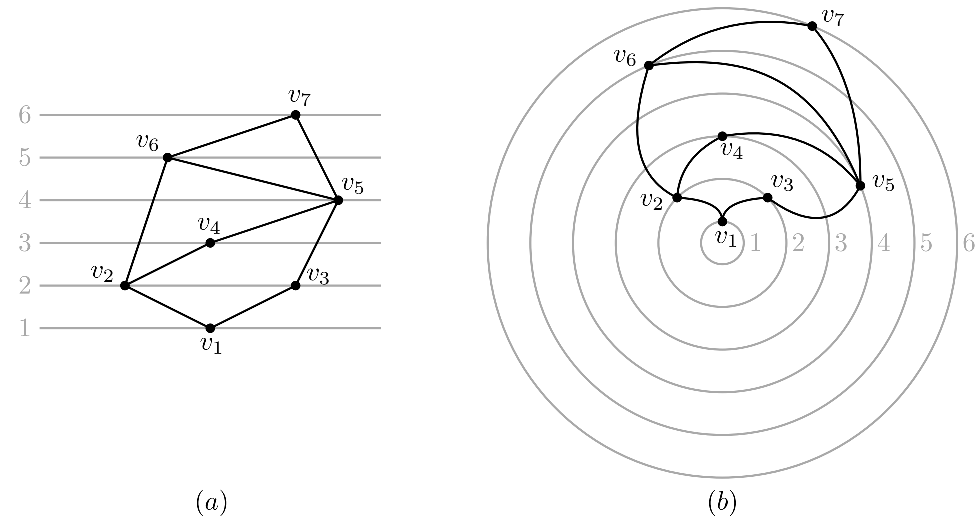


Figure 4.1: Given a leveled graph G with six levels and seven vertices. Figure (a) shows a level-planar representation, and Figure (b) is a radial-planar representation.

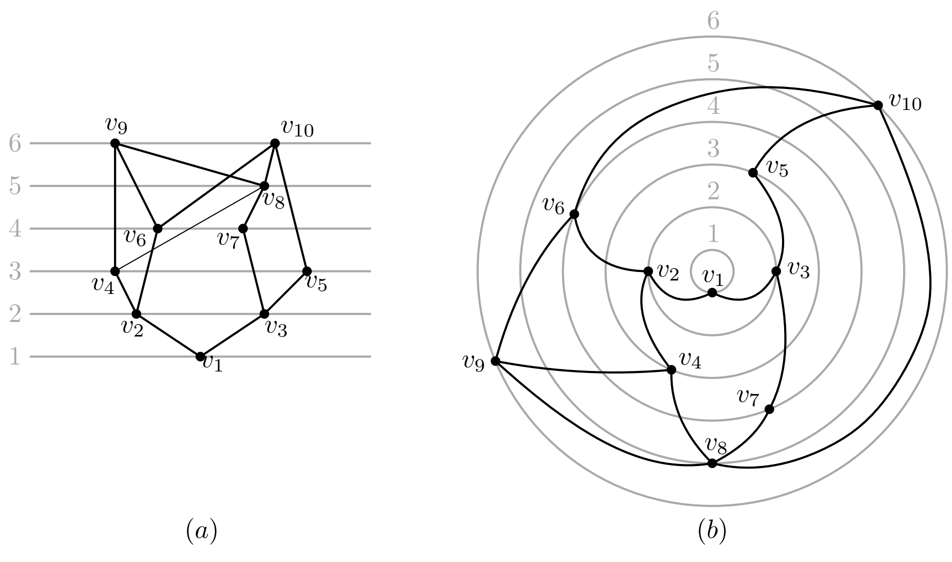


Figure 4.2: Given a leveled graph G that is not level-planar (a), but radial-planar (b).

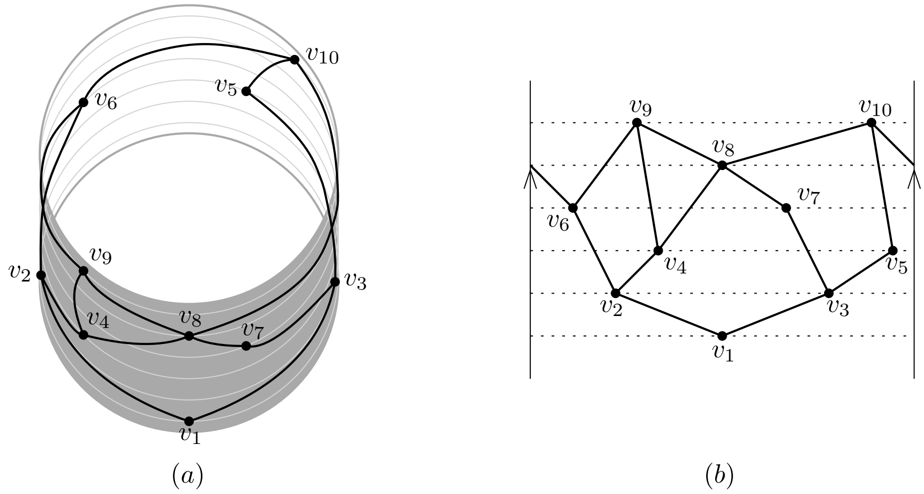


Figure 4.3: Figure (a) shows a cylindrical embedding of the graph from Figure 4.2. In Figure (b) the cylindrical embedding is represented in the flat version, which is the main form used for representation in this chapter.

drawing. The opposite sides on the left and right are identified with each other.
The notation $\ell(X)$ for some $X \subseteq \mathcal{C}$ is the interval of ℓ-coordinates of X on the interval $(0, 1)$. Assume that H is an ordered graph. Then $\max H$ means either the maximum vertex $\max V(H)$ or the maximum ℓ-coordinate $\max \ell(H)$, depending on the context. The same holds for $\min H$. Important to note is, that in a radial drawing, the maximum (minimum) ℓ-coordinate is the ℓ-coordinate of the maximum (minimum) level, since the edges do not cross these levels. Moreover, we often use for the ℓ-coordinate of a vertex v just v instead of $\ell(v)$, since in an ordered graph each vertex has a unique ℓ-coordinate.
Let u and v be vertices of a graph G. The graphs $G(u, v), G(u, v], G[u, v)$ and $G[u, v]$ are subgraphs of G restricted to the ℓ-coordinates between $\ell(u)$ and $\ell(v)$. If the bracket next to the vertex is '(' or ')' the ℓ-coordinate is excluded and with '[' or ']' it is included. Let $\mathcal{D}(G)$ be a drawing of G and G' be a subgraph of G. Then $\mathcal{D}(G')$ is the drawing of G' that is created from $\mathcal{D}(G)$ by restricting the drawing to G'.
In a radial drawing we can classify the edges into *upper* and *lower edges* at a vertex v. A upper edge e at v is an edge where $\min e = v$, and for a lower one $\max e = v$ holds. So, a vertex that only has upper edges is a *source*, and a vertex with only lower edges is a *sink*. Analogously to the classification of the edges, we can also split the rotation at a vertex v into an *upper* and *lower rotation*. The upper (lower) rotation at a vertex v is the linear order of the upper (lower) edges of v in the rotation of v. Thereby the order corresponds to the orientation of $\mathbb{S}^1$ in clockwise order. So, on the flattened cylinder both orders are read from left to right. Hence, the upper rotation is clockwise and the lower rotation counterclockwise on the cylinder.

Let G be a graph with a radial drawing $\mathcal{D}(G)$. Assume there exists a simple curve γ on the cylinder connecting $\mathbb{S}^1 \times 0$ with $\mathbb{S}^1 \times 1$ such that $\mathcal{D}(G)$ and γ do not intersect, then we can cut the cylinder along γ and get a leveled drawing of G. Hence, we can handle it as a leveled drawing or as an *x-montone drawing* by rotating it by 90 degrees. So, the levels are vertical. This is interesting, since for leveled drawings and x-monotone drawings the three versions of the Hanani-Tutte theorems exist [PT04, FPSŠ12, Bö22].
Let C be a closed curve on the cylinder. The *winding number* of C is the number of times the curve winds around the cylinder. That means, the number of times the projection of C to $\mathbb{S}^1$ passes trough a point in counterclockwise direction minus the number of windings in clockwise direction. Such a curve or a closed walk in the graph is *essential*, if the winding number is odd. The *winding number parity* of a cycle states if the winding number of this

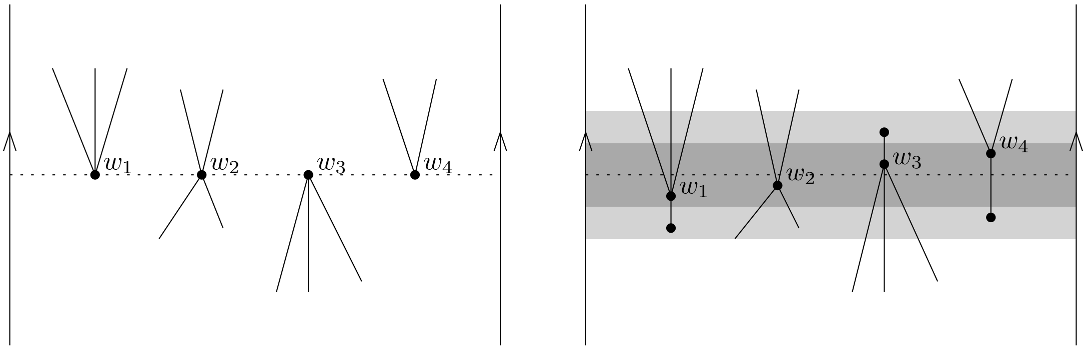


Figure 4.4: The left figure shows a level with four vertices including two sources (w_1, w_4) and a sink (w_3). The right figure shows the modification, where the original four vertices are distributed in a small neighbourhood (dark gray) and the added vertices and edges are place in the small area around (light gray).

cycle is odd or even. Hence, if the winding number parity does not change the property that a cycle is essential does not change. If a graph G contains an essential cycle, then G is *essential*.
If we have a closed curve C, we can *two-color* the complement of the curve, namely red and blue. Thereby the color changes if we cross the curve. At self-crossings the color switches twice if we walk straight over the crossing, since there we actually pass two times over the curve. So every region gets one color. If C is non-essential, the region incident to $\mathbb{S}^1 \times 0$ and the region incident to $\mathbb{S}^1 \times 1$ have the same color, let it be red. Hence, the red regions form the *exterior* of C and the blue regions form the *interior* of C.

Given a radial drawing $\mathcal{D}(G)$ of a given graph G. To change the crossing parity of a pair of edges by an non-degenerate continuous deformation of $\mathcal{D}(G)$, we perform a *vertex-edge-switch* (v, e). Thereby, we move an edge e over a vertex v that is not incident to e. Then the crossing parity between e and all edges incident to v change. A more precise definition is given by Definition 4.7.

4.2 Weak Hanani-Tutte for Radial Drawings

Given these terms, we want to state the weak Hanani-Tutte Theorem for radial planarity.

Theorem 4.1. *If a leveled graph has a radial drawing in which every two edges cross an even number of times, then it has a radial embedding with the same rotation system and leveling.*

To prove Theorem 4.1 a stronger and more powerful statement is needed. The following theorem ensures, that also the winding number parity remains.

Theorem 4.2. *If an ordered graph has an even radial drawing, then it has a radial embedding with the same ordering and the same rotation system such that the winding number parity of every cycle remains the same.*

To get Theorem 4.1 from Theorem 4.2 we use the construction shown in Figure 4.4. Thereby a leveled graph is deformed into an ordered graph, such that each vertex gets its own level. For this, assume there exists a level $l = c$ with more then one vertex. Firstly, we add to each source or sink a short edge to the side without an edge. The second endpoint of that edge is place on a new not used level below or above, respectively. This is also able for

the lowest and highest vertex, since we work on an open cylinder. Now all the vertices on level l are slightly moved such that each vertex receives its own level without passing any other level. Hence, we get an ordered graph G'. For G' exists by Theorem 4.2 a radial embedding that preserves the ordering, the rotation system and the winding number parity of every cycle. Afterwards, the added vertices and edges can be removed. Due to these added vertices and edges, there is enough room to move the original vertices back to their old level. Thus, we get a radial embedding for the original graph G with preserved rotation system and leveling.
To prove Theorem 4.2 we first introduce some tools and properties of even radial drawings.

4.2.1 Working with Even Radial Drawings

In the following, we need the concept of a *facial walk* W. This is, given a connected graph G with a rotation system, just traversing the edges according to the rotation system. At a vertex continue the walk with the consecutive edge in clockwise direction.
Moreover, if we have an ordered graph G with an ordering $v_1 < v_2 < \cdots < v_n$ we can define extrema of such facial walks. The *maximum* (*minimum*) of a facial walk W in a radial drawing of G is the vertex v_i with the highest (lowest) index i that lies on W. Analogously, a *local maximum* (*local minimum*) is a vertex v_i on a facial walk W, such that for both adjacent vertices v_j, v_k the indices j, k are smaller (larger) than i.
An *upper* (*lower*) *facial walk* of an even radial drawing is a facial walk that contains $\mathbb{S}^1 \times 1$ ($\mathbb{S}^1 \times 0$) in its interior. This definition splits the facial walks into two classes. An upper or lower facial walk is an *outer facial walk* and all other facial walks are *inner facial walks.* If a graph G has a radial embedding with only one outer face, there is also only one outer facial walk. Then the graph also has an x-monotone embedding: There exists a curve γ connecting $\mathbb{S}^1 \times 0$ and $\mathbb{S}^1 \times 1$ which does not intersect G. We cut the cylinder C along γ. By unrolling $C \setminus \gamma$ and rotating it, we get vertical levels and an x-monotone drawing. This directly introduces the following lemma.

Lemma 4.3. *In a radial embedding of a graph G exist two outer faces if and only if G contains an essential cycle.*

Proof. Assume there is only one outer face. Then we can add a curve γ to the outer face with endpoints on $\mathbb{S}^1 \times 0$ and $\mathbb{S}^1 \times 1$. The embedding of the curve γ does not intersect the embedding of G. So, every cycle is disjoint from γ and thereby the winding number of any cycle is zero. Hence, G has no essential cycle in the embedding.
If there are two outer faces, one contains $\mathbb{S}^1 \times 1$ and the other one $\mathbb{S}^1 \times 0$. Then the lower face boundary of the radial embedding is homotopic to $\mathbb{S}^1 \times 0$ and has therefore an odd winding number. Thus, G must also contain an essential cycle by induction. □

Lemma 4.4. *A cycle C in an even radial drawing is essential if and only if the two path connecting its extreme points do so in inverse order.*

Proof. Let C be a cycle with a maximum vertex v and a minimum vertex u. Let P_e be one path from u to v and P_f the second one. Both paths together form the cycle C. Thereby, P_e is connected to v by an edge e and to u by an edge e'; analogously P_f is connected to v by an edge f and to u by an edge f'. Note $e = e'$ and $f = f'$ is allowed. Assume e, e', f', f appear in C in this order as shown in Figure 4.5. Let $<_v$ be the lower rotation at v in counterclockwise direction and $<_u$ the upper rotation at u in clockwise direction. Let $e <_v f$. Then C is an essential cycle if and only if $f' <_u e'$. To prove this clam we two-color the complement of C. For this we traverse P_e. Since we have an even drawing P_e

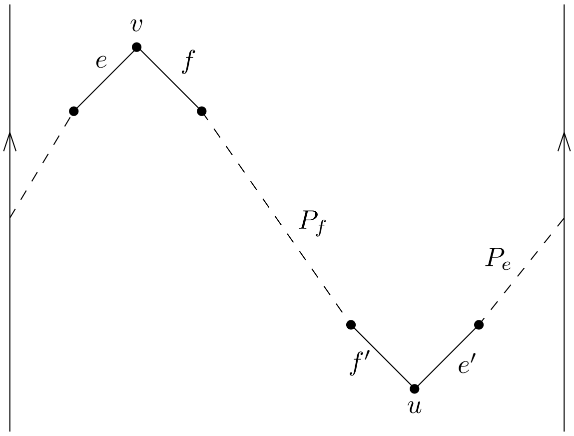


Figure 4.5: An essential cycle with maximum vertex v and minimum vertex u. The paths P_e and P_f form the cycle and they connect to u and v in inverse order.

is even. Thus, the color to the right side of P_e is the same at u and v. So the region above v has the same color as the region below u if and only if $e' <_u f'$. But then the winding number parity of cycle C is even. Hence, C is essential if and only if $f' <_u e'$. ☐

Moreover, such an essential cycle splits the graph into two parts, one above and one below. The following lemma states this behaviour.

Lemma 4.5. *Let P be a path and let C be an essential cycle, that is vertex disjoint from P in an even radial drawing of G. Then $\ell(P)$ does not contain $\ell(C)$.*

Proof. We prove this lemma by contradiction. Assume $\ell(P)$ contains $\ell(C)$. Then there exist vertices u and v with $u < \ell(C) < v$. Since P is a connected path and C is essential, there exists at least one edge e on the subpath between u and v which crosses an edge f of C an odd number of times. This is a contradiction. ☐

For the proof of Theorem 4.2, we want to work with simple facial walks. By the following lemmata, we can simplify the existing facial walks. For this, we need the concept of bounded drawings and the radial (e, v)-move.

Definition 4.6. *A given edge $e = uv$ drawn on a cylinder C is* bounded, *if $\ell(u) < \ell(p) < \ell(v)$ holds for every point p in the interior of e. A drawing of a graph G is bounded, if every edge of G is bounded.*

Definition 4.7. *Given an edge e and a vertex v not adjacent to e with $\min e < \ell(v) < \max e$. A* radial (e, v)-move *is the deformation of e such that it passes over v in a small band $\mathbb{S}^1 \times [\ell(v) - \epsilon, \ell(v) + \epsilon]$ around v as shown in Figure 4.6. Thereby, the crossing parity between e and any adjacent edge to v changes, the parity of every other pair of edges stays the same.*

Lemma 4.8. *If an ordered graph has an even bounded drawing, then it has an even radial drawing with the same rotation system.*

Proof. We want to show, that we can deform the edge $e = uv$ such that it becomes radial without changing the crossing parity between e and any other edge. Moreover, we want

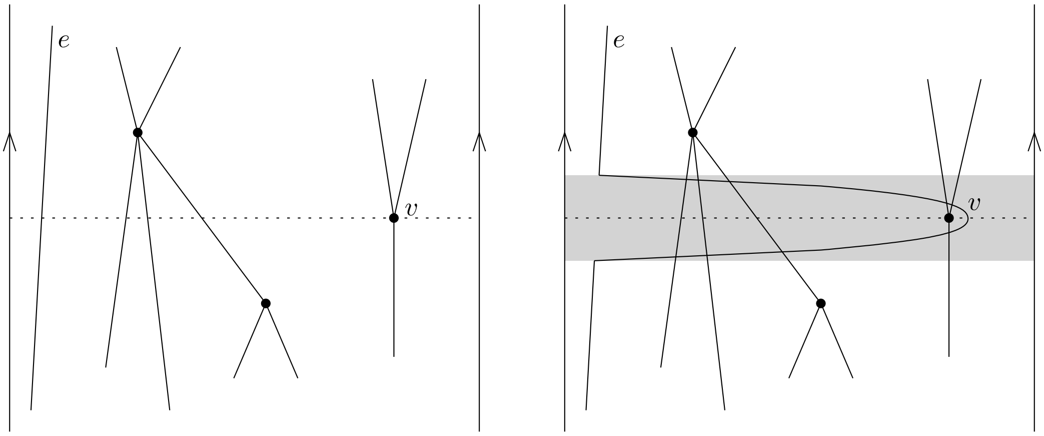


Figure 4.6: (e, v)-move in a small area $\mathbb{S}^1 \times [\ell(v) - \epsilon, \ell(v) + \epsilon]$ around v.

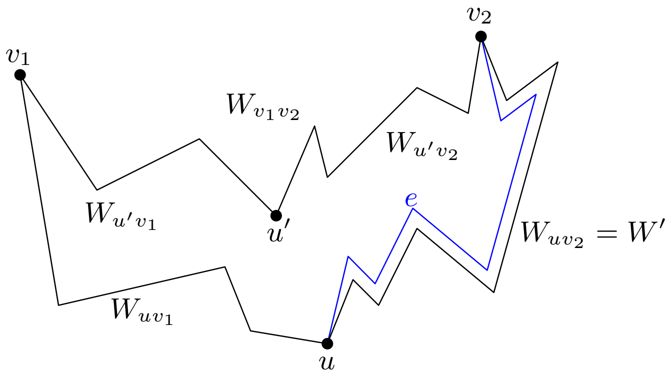


Figure 4.7: Sketch of the decomposition of a facial walk W to reduce local maxima. In blue the added edge e is shown before the redrawing step with W_{uv_2} chosen as W'.

to preserve the rotation system at u and v. We apply the necessary steps to each edge individually until the hole graph is deformed: We keep the rotation system at u and v fixed, and deform e in $\ell(e) = [u, v]$ continuously such that e is radial. For this, e passes over some vertices an odd number of times. Let us call this set of vertices S. To obtain the original crossing parities, which only changed between e and edges adjacent to vertices in S, we perform radial (e, w)-moves as described in Definition 4.7 for every vertex $w \in S$. Repeating this steps for every edge in any order until the drawing is radial, gives us the desired drawing. □

Lemma 4.9. *If an ordered graph has an even radial drawing, then we can add edges to the drawing of the graph so that the resulting drawing is still even and radial, every inner facial walk has at most two local minima and two local maxima, and each outer facial walk of each component has exactly one local minimum and one local maximum.*

Proof. We start with reducing the local maxima in an inner facial walk W. For this, assume there are at least three local maxima in W. Now we divide W into some subpaths. Let u be the vertex of W with the lowest ℓ-coordinate. Let the vertices v_1 and v_2 on W be the two local maxima with the two highest ℓ-coordinates. Thereby, also $v_1 = v_2$ is possible, since a vertex can occur multiple times on a facial walk. We split W into several subwalks, namely a walk W_{uv_1} between u and v_1, a walk $W_{v_1v_2}$ between v_1 and v_2, and a walk W_{uv_2} between u and v_2. Let u' be the local minimum in $W_{v_1v_2}$ with the lowest ℓ-coordinate, which exists by definition since v_1 and v_2 are local maxima. Then u' divides $W_{v_1v_2}$ into two

subpaths. The first walk is $W_{v_1u'}$ between v_1 and u' and the second one is $W_{v_2u'}$ between v_2 and u'. Hence, W is divided into four subwalks such that each of the subwalks has its extrema at the endpoints, as shown in Figure 4.7.
By the assumption that there are at least three local maxima, at least one subwalk contains a local maximum, which is not at an endpoint of this subwalk. Let W' be that one. To reduce the local maxima in W, we add an edge e along W' connecting the endpoints of W'. Since the endpoints of W' are its extrema, the new edge e is bounded. By the previous Lemma 4.8, we make the resulting drawing even and radial. Hence, W is replaced by two new facial walks, where one does not include the local maximum of W' and the second one does not include at least one of v_1 and v_2. Therefore, both new facial walks have fewer local extrema than W. By repeating this operation in any order, we will get at some point a drawing, where each inner facial walk contains at most two local maxima. To reduce the number of local minima, we use the same operation, but the roles of maxima and minima are swapped. Hence, we can assume, that every inner facial walk has at most two local maxima and two local minima.
Let now W be an outer facial walk. Let u be the minimum vertex of W and v the maximum vertex of W. If now either or both of the subwalks between u and v has another local extreme, we add an edge between u and v in the exterior of W. Again, we make the drawing even and radial by Lemma 4.8. Hence, the new outer facial walk has exactly one maximum and one minimum. □

The previous lemma yields in an usefull tool that each non-essential component can be placed at the position of an edge, since for these components an x-monotone embedding exists, which can be made arbitrarily small in its width.

Lemma 4.10. *If a graph G has an even radial drawing that contains no essential cycle, then G has an x-monotone embedding with the same rotation system.*

Proof. We will prove the lemma for connected graphs. If the graph is not connected, we can easily embed the x-monotone embeddings of the different components next to each other.
By Lemma 4.9, we can assume that the outer facial walk is just two edges between the extreme vertices of the graph. Note there is only one other facial walk, since G is non-essential. We can make one of these edges crossing-free without changing the rotation system with Lemma 4.11. Then cutting the cylinder along this crossing-free edge results in an even x-monotone drawing in the plane. The weak Hanani-Tutte theorem for x-monotone drawings gives us the desired embedding with unchanged rotation system, if we remove all during the process added edges [PT04]. □

4.2.2 Removing Radial Crossings

Now we want to complete the proof of Theorem 4.2. For this, we want to use the following redrawing tool, by which we can make an edge crossing-free.

Lemma 4.11. *Suppose we are given a radial drawing of a graph G (not necessarily even), and an even edge e of G. Then the edges crossing e can be redrawn inside $\mathbb{S}^1 \times \ell(e)$ to make e crossing-free and keeping the drawing radial. The redrawing does not change the rotation system, the locations of the vertices, the crossing parity of any pair of edges, or the winding number parity of any cycle.*

Proof. Let f be an edge crossing e. We cut f at every crossing with e. Since e is even, f crosses e an even number of times. Therefore, we have at each side of e an even number

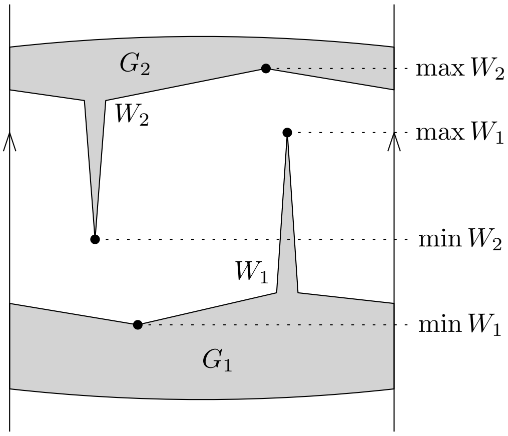


Figure 4.8: Deformation of G_1 and G_2 such that they can be embedded above each other without intersections.

of end of f. Thus, we can pair up these ends in order at each side and reconnect them locally. This yields a "curve" f with multiple components. To get back an edge f with only one component, we reconnect the components by narrow tunnels avoiding e, which is possible, since e does not separate the cylinder. Such a tunnel locally reconnect two components. Thereby, the crossing parity between f and any other edge is unchanged, since these tunnels cross another edge evenly. These operations are similar to the ones done in the proof of Theorem 3.3.
Now f may not be radial anymore. But the redrawings are done strictly between the endpoints of f. Thus, we can use the redrawing from Lemma 4.8 to make f radial. Doing this process for every edge f crossing e in any order, we remove all crossings from e. The redrawing of f does not change the rotation system, the locations of the vertices, the crossing parity of any pair of edges, or the winding number parity of any cycle. □

To finish the proof of Theorem 4.2, we firstly prove, that a counterexample to theorem has to be connected. Thereby, the proof also shows why a minimal counterexample to the strong Hanani-Tutte theorem for radial planarity also has to be connected.

Lemma 4.12. *A counterexample to Theorem 4.2 with the smallest number of vertices is connected.*

Proof. Let G be a minimal counterexample. We can assume, that G has only a restricted number of local extrema in each facial walk as given by Lemma 4.9. We now distinguish whether G has a non-essential component.
Assume there is a non-essential component H. By induction, we get an embedding of H, such that the winding number parity of cycles is preserved. Since H is non-essential, we know that the embedding is an x-monotone embedding by Lemma 4.10. Let $m_H := \min H$ and $M_H := \max H$. We can assume by Lemma 4.9 that the outer face of H is bounded by two radially drawn paths between $\ell = m_H$ and $\ell = M_H$. The given embedding of H can be deformed, such that it lies in a small neighbourhood of a curve $\mathcal{C}$ with $\ell(C) = [m_H, M_H]$. This can be done independently for each non-essential component. Thus, if every component is non-essential, we combine the different embeddings to one embedding of G by placing them next to each other. Hence, we can assume that G contains at least one essential component.
Let G_* be the subgraph containing all essential components of G. Again, by induction we get a radial embedding of G_*. Let E' be the set of edges e in G_* such that $M_H \in \ell(e)$.

Then these edges divide G_* into two subgraphs G_*^+ and G_*^- induced by the vertex sets $\{v \in V(G_*) : v > M_H\}$ and $\{v \in V(G_*) : v > M_H\}$, respectively.
Assume for the upper outer face of G_*^- that it lies completely above $\ell = m_H$. Then the boundary of that face contains an essential facial walk W_U. This walk W_U lies between m_H and M_H. But this is a contradiction to Lemma 4.5, since $\ell(H)$ would contain $\ell(W_U)$. Therefore, the upper outer face of G_*^- intersects $\ell = m_H$. Moreover, every face of G_* lying in the upper outer face of G_*^- intersects $\ell = M_H$. Hence, in G_* there exists a face f intersecting both m_H and M_H. In f is a radially drawn path P_f connecting its maximum and minimum by Lemma 4.9. Then H can be embedded as above along P_f, and also every other non-essential component in the same way. Hence, we can assume that all components of G are essential.
Let G_1 be the component of G, that contains v_1 and let $G_2 = G \setminus G_1$ be the rest of the graph. By induction we get embeddings for G_1 and G_2. Let W_1 be the upper facial walk of G_1 and W_2 the lower facial walk of G_2. Since W_1 and W_2 are essential and $\min W_2 > v_1$, Lemma 4.5 implies that $\max W_2 > \max W_1$ and $\min W_2 > \min W_1$. By Lemma 4.9 both W_1 and W_2 consists of two radially drawn curves. Then we deform the radial embedding of G_1 such that it lies, except for a small strip near $\ell = \max W_1$, in a small neighbourhood around a radial curve in the region between $\min W_1$ and $\max W_1$. Analogously, we deform the embedding of G_2 such that it lies, except for a small strip near $\ell = \min W_2$, in a small neighbourhood around a radial curve in the region between $\min W_2$ and $\max W_2$. See Figure 4.8 for an illustration. Now, these two embeddings can easily be placed above each other such that they do not intersect. Hence, we get an embedding of G satisfying Theorem 4.2. □

Given these lemmata we now prove Theorem 4.2:

Theorem 4.2. *If an ordered graph has an even radial drawing, then it has a radial embedding with the same ordering and the same rotation system such that the winding number parity of every cycle remains the same.*

Proof of Theorem 4.2. Assume we have a counterexample to Theorem 4.2. Moreover, we can assume that the counterexample does not have an inner facial walk with just two edges. Since we can remove one of the edges, get an embedding of the rest by induction and embed the removed edge next to the other one.
We pick the counterexample G such that it has the fewest vertices and among these the largest number of edges. By the Lemmata 4.12 and 4.9 we know that G is connected, every inner facial walk contains at most two local minima and maxima and every outer facial walk contains at most one local minimum and maximum.
Let D_i be an even radial drawing of G restricted to $\mathbb{S}^1 \times [\ell(v_1), \ell(v_i)]$. The goal is to stepwise prove for $1 \leq i \leq n$ that D_i is a crossing-free radial drawing. Then D_n is the desired drawing. Let D_1 be the given drawing of G. We assume that D_i for $i < n$ is given and show how to redraw it to obtain D_{i+1}. We distinguish whether v_{i+1} is a source. We start with the case that v_{i+1} is not a source.

Vertex v_{i+1} is not a source. Then v_{i+1} has at least one lower edge. Let e be such an edge. By Lemma 4.11, we make e crossing-free without adding crossings in the region $\mathbb{S}^1 \times [\ell(v_1), \ell(v_i)]$, which is already free of crossings by assumption. The setting is shown in Figure 4.9 (a).
We define π_i to be the cyclic order of elements of G intersecting the circle $\mathbb{S}^1 \times \ell(v_i)$. Thereby, we replace v_i (the only vertex on that level) by the order of the edges in its upper rotation.
We claim, that the lower edges of v_{i+1} have to appear consecutive in π_i. Assume otherwise,

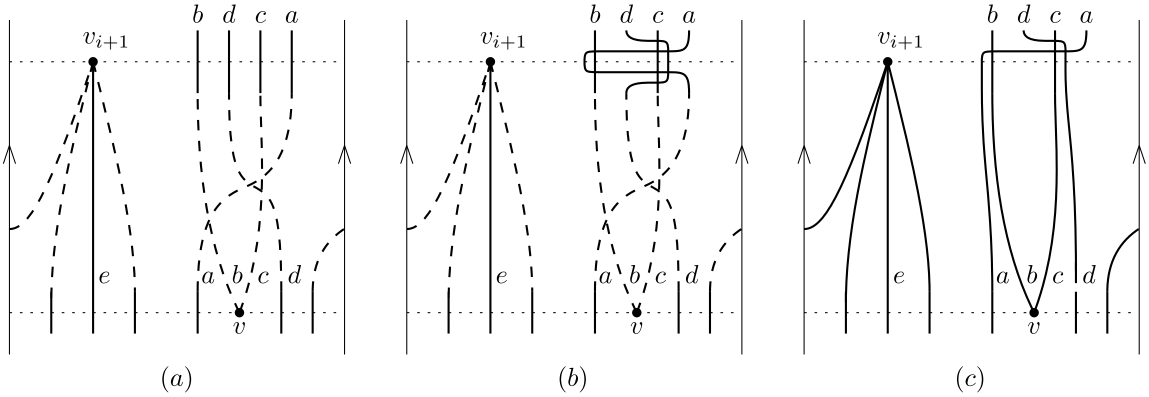


Figure 4.9: The steps of redrawing D_i to obtain D_{i+1} if v_{i+1} is a source. Dashed edges can contain crossings.

that there is a subsequence a, b, c, d of edges in π_i, where a and c are incident to v_{i+1} and b and d are not. But then b or d have to cross a or c an odd number of times. There cannot be another crossing in the region $\mathbb{S}^1 \times [\ell(v_1), \ell(v_i)]$, since there all edges cross evenly. That is a contradiction to the condition that we have an even drawing.
To remove crossings in the region $\mathbb{S}^1 \times [\ell(v_i), \ell(v_{i+1})]$, we can change the order the edges intersect the circle $\mathbb{S}^1 \times \ell(v_{i+1})$. For this, we continuously deform the edges in the region $(\mathbb{S}^1 \times [\ell(v_{i+1}) - \epsilon, \ell(v_{i+1}) + \epsilon]) \setminus (e \cup v_{i+1})$ as shown in Figure 4.9 (b). Due to the deformation, no odd crossing is added, since the modified region does not contain a vertex. Thus, crossings are added pairwise. With this operation we change the order of edges intersecting $(\mathbb{S}^1 \times \ell(v_{i+1})) \setminus v_{i+1}$ such that they match the ordering from π_i, where the edges incident to v_{i+1} are removed. This gives us a drawing, where the edges in the region $(\mathbb{S}^1 \times \ell(v_{i+1})) \setminus v_{i+1}$ cross evenly and the orderings at the top and bottom match. So, it is easy to redraw the edges in that region without any crossings, resulting in the desired drawing D_{i+1}, as shown in Figure 4.9 (c).

Vertex v_{i+1} is a source. Let W be the facial walk, such that the two edges reaching v_{i+1} are consecutive in clockwise order and v_{i+1} occurs as a local minimum. Then W has two local maxima (possibly the same vertex), because v_{i+1} is a source. Let z be the maximum vertex, y the second local maximum and u the minimum of W. The facial walk W contains by Lemma 4.9 four radially-drawn paths: a path P between u and y, a path Q between y and v_{i+1}, a path T between v_{i+1} and z, and a path between z and u. Moreover, we make P crossing-free by repeatedly applying Lemma 4.11 to each edge of P individually. The vertex v_{i+1} together with the path P split the circle $\mathbb{S}^1 \times \ell(v_{i+1})$ into two parts. Let S be the part that reaches P from the interior of W. Analogously, we define T to be the part of the circle $\mathbb{S}^1 \times y$, that lies between y and R and reaches R from the exterior of W. The setting is shown in Figure 4.10.
If now S is crossing-free, we can add an edge along S and P between u and v_{i+1}. But then v_{i+1} is not a source any more. Hence, we can apply induction and are done.
So, we assume that S is not crossing-free. However, we want to make S crossing-free, while keeping the drawing even and without crossing P. The curve formed by S, Q and the part of P between y and S is simple and closed. Let V_S be the set of vertices contained in its interior (v_{i+1} and y are not part of the interior). Similarly, we define a simple closed curve formed by T, Q and the part of R between v_{i+1} and T. Then let V_T be the set of vertices on this curve and its interior. We pick the side of the curve as interior that does not contain V_S. So, V_S and V_T are disjoint. This is also illustrated in Figure 4.10.
Let E_S be the set of edges, where one endpoint is in V_S and the other endpoint is in $V \setminus (V_S \cup \{v_{i+1}, y\})$. Then each such edge have to cross S oddly, since the edges of P are

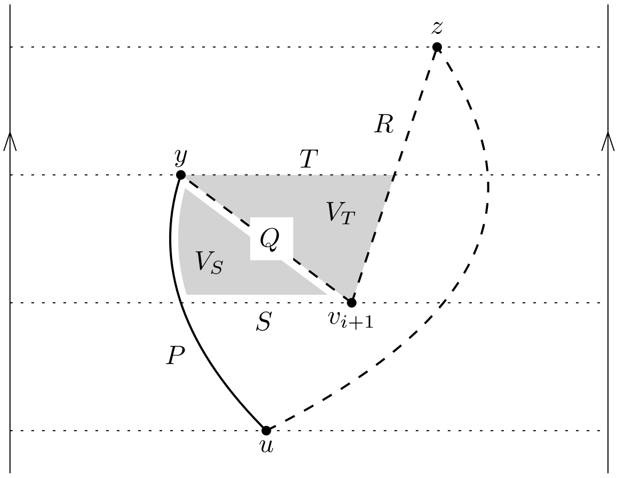


Figure 4.10: The facial walk W where v_{i+1} is a source. The dotted lines represent different levels and the dashed curves may be crossed.The grey regions represent the vertex sets V_S and V_T.

crossing-free and the edge of Q are even. Thus, the second endpoint lies below $\ell = \ell(S)$. Analogously, we define E_T to be the set of edges, where exactly one endpoint is in V_T. Then such an edge $e \in E_T$ crosses either T oddly or e crosses R below $\ell = \ell(T)$ oddly and a second time above $\ell = \ell(T)$ oddly, since Q is even. In both cases, the second endpoint of e lies above $\ell = \ell(T)$. Hence, E_S and E_T are disjoint and share no incident vertex.
Let us consider a vertex w and an edge e with $w \in \ell(e)$. Then we perform a radial (e, w)-move without crossing P if either $e \in E_S$ and $w \in V_T$ or $e \in E_T$ and $w \in V_S$. This yields still to an even drawing, since the only crossing parity that may change is between edges $e_S \in E_S$ and $e_T \in E_T$. But for these pairs of edges the crossing parity changes for each endpoint in $\ell(e_S) \cap \ell(e_T)$ which is in $V_S \cup V_T$. So, if $\ell(e_S) \cap \ell(e_T) \neq \emptyset$, then there are exactly two endpoints of e_S and e_T, for which an (e, w)-move is performed. Hence, the crossing parity between e_S and e_T remains even.
The (e, w)-moves with $w = v_{i+1}$ move all crossings off of S and above v_{i+1}. Since no other (e, w)-move changes something in the region around $\ell = \ell(v_{i+1})$, S is crossing-free, and we can argue as above, where we added an edge between v_{i+1} and u along S and P.
The rotation system is preserved by all operations, since no ordering at a vertex is changed. By Lemma 4.4, the winding number parity is unchanged, because the rotation system is the same. □

4.3 Strong Hanani-Tutte for Radial Drawings

Given the weak Hanani-Tutte Theorem for radial drawings the next step is to prove the strong version of the theorem. We want to use the weak Hanani-Tutte theorem as a basis for finding a minimal counter example. Therefore, a minimal counter example must contain at least one pair of adjacent edges crossing oddly.

Theorem 4.13. *If a leveled graph, possibly with multiple edges but without loops, has a radial drawing in which every two independent edges cross an even number of times, then it has a radial embedding.*

The proof of the weak version uses that the parity of the winding number of cycles is preserved in even radial drawings. But if we only have an independently even radial

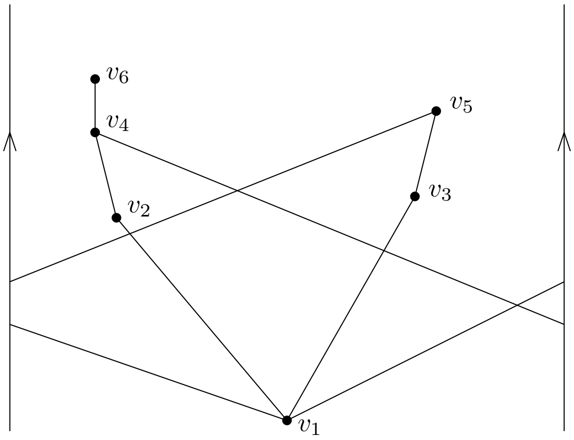


Figure 4.11: An independently even radial drawing with two essential cycles $v_1v_2v_4$ and $v_1v_3v_5$. To obtain a radial embedding the winding number parity of at least one cycle has to change.

drawing, this property is not preserved. Figure 4.11 shows a counterexample, where the winding number parity cannot be preserved to obtain a radial embedding.

Therefore, we need other concepts to prove the strong version. Suppose we have a graph G and two radial drawings $\mathcal{D}_1$ and $\mathcal{D}_2$ of G. Then $\mathcal{D}_2$ *is supported by* $\mathcal{D}_1$ if for every essential cycle C_2 in $\mathcal{D}_2$ there exists an essential cycle C_1 in $\mathcal{D}_1$ such that $[\min C_1, \max C_1] \subseteq [\min C_2, \max C_2]$. That means, that for every essential cycle C_2 in $\mathcal{D}_2$, there must be an essential cycle C_1 in $\mathcal{D}_1$ such that the height of C_1 on the cylinder is contained in the height of C_2. Thus, the height for the essential cycles in $\mathcal{D}_2$ cannot be reduced below the height in $\mathcal{D}_1$. Note that there may be more essential cycles in $\mathcal{D}_2$ than in $\mathcal{D}_1$.
The second definition is that a drawing is *weakly essential*. Given an ordered connected graph G, a radial drawing of G is weakly essential, if every essential cycle contains the first vertex v_1 or the last vertex v_n. Now again we use a stronger version to prove Theorem 4.13.

Theorem 4.14. *Let G be an ordered graph. Suppose that G has an independently even radial drawing. Then G has a radial embedding. Moreover, (i) if the given drawing of G is weakly essential, then G has an x-monotone embedding; and (ii) the new radial embedding is supported by the original drawing.*

Theorem 4.13 follows from Theorem 4.14 by an equivalent construction as we used in Section 4.2 to prove Theorem 4.1. Thereby, we convert the given graph into an ordered graph, apply Theorem 4.14 and then rebuild the original graph.

4.3.1 Working with Independently Even Radial Drawings

In the following we state some properties of (independently) even radial drawings, which will be helpful in the proof of Theorem 4.14. Some of the results are extended results from Section 4.2.

Lemma 4.15. *If an ordered graph has an (independently) even bounded drawing, then it has an (independently) even radial drawing with the same rotation system which preserves whether cycles are essential or non-essential.*

Proof. We want to show, that we can deform the edge $e = uv$ without changing the crossing parity between e and any other edge, preserve the rotation systems at u and v and do not

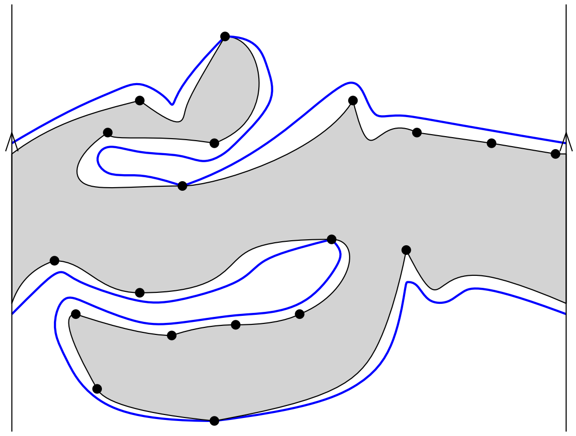


Figure 4.12: A graph with added edges (blue) in the outer faces to obtain 2-faces.

change the winding number of any cycle. Then apply the necessary steps to each edge individually until the hole graph is deformed. So, we keep the rotation system at u and v fixed, and deform e in $\ell(e) = [u, v]$ continuously such that e is radial. For this, e passes over some vertices an odd number of times. Let us call this set of vertices S. To obtain the original crossing parities, which only changed between e and edges incident to vertices in S, we perform radial (e, w)-moves as described in Definition 4.7 for every $w \in S$. By repeating these steps for every edge in any order until the drawing is radial, we get the desired drawing. $\square$

We know by the lemma above, that it is enough to find a bounded drawing. Moreover, we can use this lemma, to reduce the local extrema in each face as stated by the following corollary.

Corollary 4.16. *In a radial embedding of a connected ordered graph G we can subdivide any face f by adding an edge joining its maximum vertex with its minimum one while keeping the embedding radial. If f is an outer face, we can subdivide it by adding at most two edges so that the new outer face contains exactly one local minimum and maximum.*

Proof. For the first part of the corollary we take a face f and add a new bounded edge e between the minimum and maximum of f along the boundary of f. Since the edge is bounded, we apply Lemma 4.15 to get a radial drawing of e. Moreover, e is an even edge, since it is placed along the boundary without crossing any other edge. Thus, we use the weak Hanani-Tutte Theorem (Theorem 4.1) to get a radial embedding.
For the second part, we assume f is an outer face. The facial walk W of f can be split into two subwalks between the minimum and maximum. Along each subwalk we can add an edge in f as shown in Figure 4.12. If the subwalk is just a single edge, we use this edge. The two edges form a new boundary walk W' bounding a 2-face f'. This new outer face f' replaces the old outer face f. By applying Lemma 4.15 and Theorem 4.2 we obtain a radial embedding for the new edges. Therefore, the rotation system is unchanged, and since G is connected, W' also bounds the new face f'_*. Moreover, the winding number of W' is not changed by these operations. Thus, by Lemma 4.4 f'_* is essential if and only if f' was essential and the unchanged rotation system makes sure the face f'_* is an outer face. The new outer face has only two vertices. Hence, only one local minimum and maximum can exist. $\square$

Lemma 4.17. *Let P be a path and let C be an essential cycle, vertex disjoint from P, in an independently even drawing of a graph. Then $\ell(P)$ does not contain $\ell(C)$.*

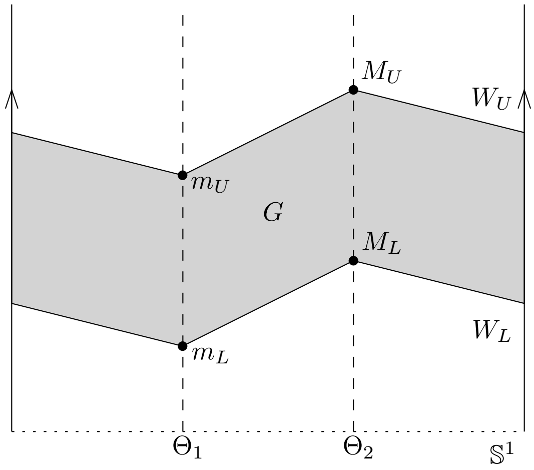


Figure 4.13: A graph G (gray) with upper and lower face bounded by essential 2-cycles with aligned minima and maxima.

Proof. We prove this lemma by contradiction. Assume $\ell(P)$ contains $\ell(C)$. Then there exist vertices u and v with $u < \ell(C) < v$. Since P is a connected path and C is essential, there is at least one edge e on the subpath between u and v, which crosses an edge f of C an odd number of times. Given that P and C are vertex disjoint, e and f are independent edges. But this contradicts that the given drawing is independently even. □

The next part describes how we combine embeddings of different components. Let H be a graph with an independently even radial drawing without essential cycles, then H has an x-monotone embedding. This follows by Lemma 4.22, where we prove a slightly stronger statement. Moreover, this x-monotone embedding can be reduced in width as required. Thus, we can embed H arbitrarily close next to a radial curve e with $\ell(H) \subseteq \ell(e)$. This is called a “skinny” embedding.

Observation 4.18. *If an ordered graph H has an x-monotone embedding and e is a radial curve on the cylinder with $\ell(H) \subseteq \ell(e)$, then H has an embedding on the cylinder that lies arbitrarily close to e.*

If we have a radially-embedded graph with essential cycles, we find a level-preserving deformation, such that the maxima and minima of the outer faces can be aligned at any distinct angles Θ_1 and Θ_2 on $\mathbb{S}^1$, as shown in Figure 4.13.
Let Θ_1 and Θ_2 be angles on $\mathbb{S}^1$ and m and M be two height in $[0, 1]$, then $\gamma(\Theta_1, m, \Theta_2, M)$ is an essential 2-curve connecting the two points (Θ_1, m) and (Θ_2, M) by two edges, such that the curve is an essential cycle.

Lemma 4.19. *Suppose an ordered graph G containing essential cycles is radially embedded. Let m_L and M_L be the minimum and maximum of the lower face boundary and let m_U and M_U be the minimum and the maximum of the upper face boundary. Then for any distinct Θ_1, Θ_2 on $\mathbb{S}^1$, there is a radial embedding of G that lies between the curves $\gamma(\Theta_1, m_L, \Theta_2, M_L)$ and $\gamma(\Theta_1, m_U, \Theta_2, M_U)$.*

Proof. Since G contains an essential cycle, by Lemma 4.3 the graph G has a separate upper and lower outer face. By Corollary 4.16 we can add edges, such that we have upper and lower outer 2-faces with radially-drawn edges. We can deform the edges of these outer 2-faces such that the endpoints stay on their level and the upper face matches the curve $\gamma(\Theta_1, m_U, \Theta_2, M_U)$ and the lower face matches the curve $\gamma(\Theta_1, m_L, \Theta_2, M_L)$. After removing the added edges, we get an embedding of G between these to certain curves. □

4.3.2 Weakly Essential Drawings

Given a vertex v, such that two consecutive edges e and f in the rotation at v cross an odd number of times. We can remove this odd crossing, by flipping e and f in the rotation at v. This operation adds a crossing between e and f such that e and f cross evenly. Thereby, both e and f are either in the upper or in the lower rotation of v. The idea is to use the proof of the strong Hanani-Tutte theorem for x-monotone drawings as a reference and construct the proof for radial drawings similarly [FPSŠ12, Theorem 3.1]. In that setting are to possible obstacles, if we are not able to make incident edges at a vertex v cross evenly by flipping. Either there is a component H of $G\{v, w\}$ for some $w \in V(G)$ such that $v \leq \min H < \max H \leq w$ or a multiple edge vw. We handle both cases by reducing them to the weak Hanani-Tutte theorem for x-monotone drawings [FPSŠ12, Theorem 2.11]. But this approach does not work for the radial setting. There can exist a vertex v such that its incident edges can not be made crossing evenly by flipping operations and there is neither a component H nor a multiple edge vw. This can only occur at the first or the last vertex of the corresponding ordered graph as we show in Lemma 4.22. For this, we define a new graph G'.

Definition 4.20. *Given an ordered graph G with vertices $v_1 < \cdots < v_n$ and without the edge v_1v_n. Define G' to be the ordered graph obtained from G by removing v_1 and v_n and replacing the edges to these vertices. Let w_i with $i \in 1, \ldots, k$ be the adjacent vertices of v_1 in G. For G' replace each edge v_1w_i by a new edge $v_i'w_i$ preserving the crossings appearing in G. The vertex v_i' is a new degree-1 endpoint for each new edge. Analogously, let w_j' with $j \in 1, \ldots, l$ be the adjacent vertices of v_n in G. For G' replace each edge v_nw_j' by a new edge $v_j''w_j'$ such that the crossings appearing in G are preserved. The vertex v_j'' is new degree-1 endpoint for each new edge.*

Lemma 4.21. *If G is a connected ordered graph with an (independently) even radial drawing $\mathcal{D}(G)$, then G' has an (independently) even radial drawing $\mathcal{D}'(G')$ such that $\mathcal{D}'(G \setminus \{v_1, v_n\}) = \mathcal{D}(G \setminus \{v_1, v_n\})$ and $\mathcal{D}'(G')$ is supported by $\mathcal{D}(G)$.*

Proof. Let us construct G' as in Definition 4.20. Then the originally oddly crossing pairs at v_1 in $\mathcal{D}(G)$ except of multiple edges become independent edge crossings. To remove these crossings, we apply some radial (e, v)-moves. Let w_i be the upper endpoint of the edge incident to v_i' for $1 \leq i \leq k$. If two edges $v_i'w_i$ and $v_j'w_j$ with $i < j$ cross an odd number of times, we perform a radial $(v_i'w_i, v_j')$-move to add one crossing between $v_i'w_i$ and $v_j'w_j$ such that they cross evenly. We apply this procedure to all pairs. So, for each $j \in 1, \ldots, k$ we perform for each edge $v_i'w_i$ that crosses $v_j'w_j$ oddly a radial $(v_i'w_i, v_j')$-move. Afterwards all $v_i'w_i, v_j'w_j$-pairs cross evenly.
Similarly, we handle v_n with the new endpoints $v_1'' < \cdots < v_l''$. Thus, we get a drawing $\mathcal{D}'(G')$ of G' with $\mathcal{D}'(G \setminus \{v_1, v_n\}) = \mathcal{D}(G \setminus \{v_1, v_n\})$. Moreover, every essential cycle of $\mathcal{D}'(G')$ is also essential in $\mathcal{D}(G)$ such that, $\mathcal{D}'(G')$ is supported by $\mathcal{D}(G)$. □

With Lemma 4.21 we can prove part (i) of Theorem 4.14.

Lemma 4.22. *Suppose that G has an independently even radial drawing $\mathcal{D}(G)$ that is weakly essential. Then G has an x-monotone embedding.*

Proof. By the previous Lemma 4.21 we get an independently even radial drawing $\mathcal{D}'(G')$. Given that $\mathcal{D}(G)$ is weakly essential and $\mathcal{D}'(G')$ is supported by $\mathcal{D}(G)$, we know by the construction of $\mathcal{D}'(G')$ that the only possible existing essential cycles in $\mathcal{D}(G)$ (containing v_1 or v_n) are erased in $\mathcal{D}'(G')$. Hence, $\mathcal{D}'(G')$ has no essential cycles.

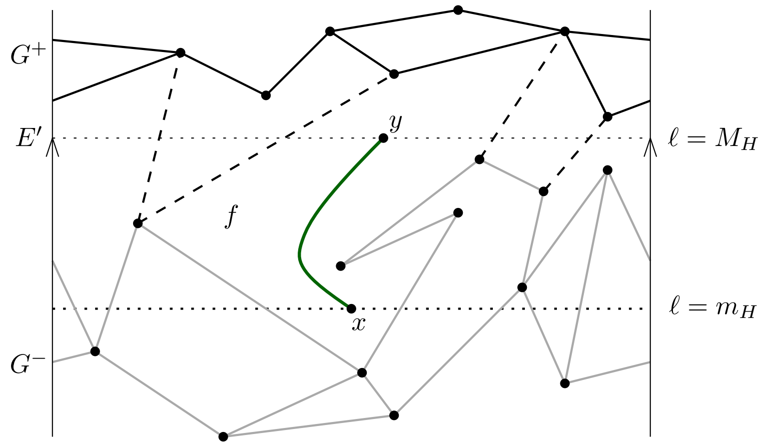


Figure 4.14: The set of edges E' (dashed) and the induced graphs G^- (gray edges) and G^+ (black edges); the green radial curve xy in face f represents the place for $m_H M_H$ and H respectively.

We add new vertices v, v' with $v < \min G' \leq \max G' < v'$ and a radial edge $e = vv'$ such that e does not contain any vertex of G'. Define E' to be the set of edges of G' that crosses e an odd number of times. Moreover, E' is an edge-cut of G' and by performing radial (e, w)-moves for all vertices w on one side of the cut E', we make e even. By Lemma 4.11 based on the graph $(V(G') \cup \{v, v'\}, E(G') \cup \{e\})$, we can make e crossing free. Then we use e as a cut line and deform the cylinder C into a subset of the plane. After rotating the radial drawing, it becomes x-monotone. With the strong version of the Hanani-Tutte theorem for x-monotone drawings we get an x-monotone embedding of G' [FPSŠ12]. To get a drawing of G, we combine the vertices v'_i for $1 \leq i \leq k$ to the vertex v_1 and v''_j for $1 \leq j \leq l$ to v_n. This is done, such that we do not add crossings. Hence, we get an x-monotone embedding of G. □

4.3.3 Components of a Minimal Counterexample

To prove part *(ii)* of Theorem 4.14, we work with a minimal counterexample. Let G be a minimal counterexample with the fewest number of vertices and among them one with the fewest edges. Let $\mathcal{D}(G)$ be an independently even radial drawing of G. Then there is no radial embedding of G that is supported by $\mathcal{D}(G)$. Moreover, $\mathcal{D}(G)$ is not weakly essential by the previos Lemma 4.22.

Lemma 4.23. *A minimal counterexample does not contain multiple edges.*

Proof. Assume the graph G for a minimal counterexample contains multiple edges e, e' with endpoints u and v. By induction, we get a drawing $\mathcal{D}(G - e)$ of $G - e$. We add e along e' without crossings to the drawing to receive an embedding of G. This embedding is supported by $\mathcal{D}(G - e)$, since for every essential cycle C, that contains e, the cycle $C' = C - e + e'$ is also essential and in $\mathcal{D}(G - e)$ and moreover $\ell(C') = \ell(C)$. □

Lemma 4.24. *G is connected.*

Proof. Assume G is not connected. The first case is, that there exists a non-essential component H of G with $m_H = \min V(H)$ and $M_H = \max V(H)$. Then we embed $G' = G \setminus H$ by induction. We want to add a crossing-free edge $m_H M_H$ to the embedding. By Observation 4.18, this edge is enough such that H can be embedded around the area of

$m_H M_H$.
For this, we divide G' into a set of edges E' and two embedded subgraphs G^- and G^+. E' is the set of edges e with $\min e < M_H < \max e$; the set of vertices $\{v \in V(G') : v < M_H\}$ induces G^- and $\{v \in V(G') : v > M_H\}$ induces G^+ as shown in Figure 4.14. By definition, the upper face of G^- contains $\ell = M_H$ and it must also intersect $\ell = m_H$. If not, the upper boundary of G^- contains between m_H and M_H an essential cycle, which contradicts part (*ii*) of Theorem 4.14 with Lemma 4.17 and H. Define a point x with $\ell(x) = m_H$ in the upper boundary of G^- such that x does not intersect with E' and the face f containing x must intersect with $\ell = M_H$. Thus, we can add a curve xy from $\ell = m_H$ to $\ell = M_H$, which can be made radial by Corollary 4.16. By Observation 4.18 this edge can be replaced by $m_H M_H$ or H. This results in an embedding of G and satisfies part (*ii*) of Theorem 4.14, because the previous operations do not remove or add any essential cycle.
The second case is, that every component of G is essential. Let H be the component of G with $\max H = \max G$. We again embed $G' = G \setminus H$ by induction. By the same argumentation as above, the upper face of G' must contain $\ell = \max H$ and intersect $\ell = \min H$, since otherwise there exists an essential cycle between m_H and M_H contradicting part (*ii*) of Theorem 4.14 with H and Lemma 4.17. Thus, the minimum m_U of the upper boundary of G' is below the minimum of H. G' can be embedded on a cylinder below the curve $\gamma(0, m_U, \pi, \max G')$ by Lemma 4.19. The component H can also be embedded by induction such that the maximum M_L of the lower boundary of H satisfies $\max G' < M_L$. Again by Lemma 4.19 the embedding of H lies strictly above the curve $\gamma(0, \min H, \pi, M_L)$. These two embeddings of G' and H do not intersect, since $m_U < \min H$ and $\max G' < M_L$ and so, the two curves lie with a gap above each other. This is an embedding of G satisfying part (*ii*) of Theorem 4.14 since all essential cycles lie in a component which was drawn by induction. □

Assume we have a vertex v of G. Let B be a component of $G \setminus v$ with $\min B > v$. Then G contains at least one edge from v to a vertex in B, since G is connected by Lemma 4.24.

Lemma 4.25. *Let v be a vertex and B be a component of $G \setminus v$ with $\min B > v$. Then either $|V(B)| = 1$ or B is essential in $D(G)$.*

Proof. We prove the statement by contradiction. For this assume that B is free of essential cycles and $|V(B)| \neq 1$. We define B' to be the induced subgraph of $V(B) \cup \{v\}$. Since B do not contain any essential cycle, we can apply Lemma 4.22 to get an x-monotone embedding $\mathcal{E}(B')$ of B'. Let vPw be a path in B' from v to $w = \max B$. We replace B' by a single curve e from v to w to get a graph G'. The drawing $\mathcal{D}(G')$ of G' is almost the same as $\mathcal{D}(G)$ only $\mathcal{D}(e) = \mathcal{D}(P)$. For this, the vertices of P are omitted to get the drawing $\mathcal{D}(e)$. The curve e is maybe not radial. So, $D(G')$ does not have to be radial. But since e is bounded and independently even, we can apply Lemma 4.15 to get an independently even drawing $\mathcal{D}'(G')$ of G'. By induction, we get a radial embedding $\mathcal{E}(G')$ of G'. To complete the embedding to an embedding for G, we use Observation 4.18 to replace e in $\mathcal{E}(G')$ by a "skinny" version of $\mathcal{E}(B')$ intersecting in v and w. So, the created embedding of G is supported by $\mathcal{D}(G)$ contradicting the assumption of a minimal counterexample. □

Lemma 4.26. *Let v be a vertex and B a component of $G \setminus v$ with $\min B > v$. If B is essential in $D(G)$, then $v = v_1$.*

Proof. Let G be a graph. Assume B is essential and $v \neq v_1$. We consider the components of $G \setminus v$. Let G_1 be the union of the components H with $\min H > v$ and the union of the other components be G_2. Both subgraphs are non empty, since B is part of G_1 and the vertex

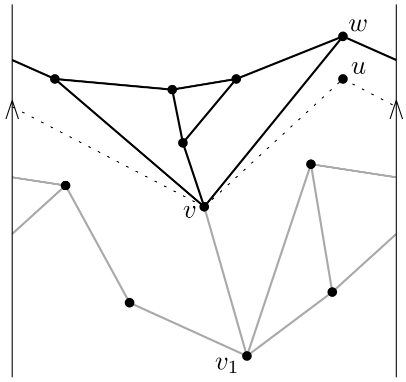


Figure 4.15: G'_1 in black and $G'_2 \setminus w$ in gray. Vertex u of G''_2 directly below $\max W_1$ and dotted the upper outer 2-face of G''_2.

$v_1 \neq v$ is in G_2. Let G'_1 be the subgraph of G induced by $V(G_1) \cup \{v\}$ and G'_2 the union of the subgraph induced by $V(G_2) \cup \{v\}$ and an edge e between v and $w := \max G'_1 = \max G$. In G_1 exists a path vPw connecting v and w. For this path, we can argue as in the proof of Lemma 4.25, where the inserted edge e is affected, that there exists an independently even radial drawing of G'_2. Moreover, G'_2 is not equal to G, since G'_1 contains an essential cycle, because B is in G. Figure 4.15 shows an example.
By the minimality of G, there are radial embeddings $\mathcal{E}_1(G'_1)$ and $\mathcal{E}_2(G'_2)$ of G'_1 and G'_2. Each embedding is supported by the existing drawings of G'_1 and G'_2, respectively. We distinguish two cases, whether $\mathcal{E}_1(G'_1)$ is essential. If $\mathcal{E}_1(G'_1)$ is non-essential, we insert a "skinny" embedding of $\mathcal{E}_1(G_1)$ along the embedding of e in $\mathcal{E}_2(G'_2)$ intersecting in v and w. Thus, we get an radial embedding of G.
The second case is that $\mathcal{E}_1(G'_1)$ is essential. Then the lower facial walk W_1 of G'_1 in $\mathcal{E}_1(G'_1)$ is essential. Moreover, $\max W_1 > \max(G'_2 \setminus w)$. If this claim would be wrong, there must be a vertex y in $G'_2 \setminus w$ with $y > \max W_1$. But every component H of G_2 has at least one vertex x with $x < v$ by definition. Thus, there must be a path xPy from x to y in $G'_2 \setminus w$. But this path and W_1 contradict Lemma 4.17. So, the claim that $\max W_1 > \max(G'_2 \setminus w)$ holds. To finish the proof, we define G''_2 from G'_2 in three steps. First, we subdivide e in G'_2 by adding a vertex u just below $\max(G'_1)$ and above $\max(G'_2 \setminus w)$. Let e' be the new edge between v and u. Secondly, we remove w and thus also uw from G'_2. Finally, we add a second edge e'' between v and u such that $ve'ue''v$ form an essential 2-cycle. The edge e'' can be added to the embedding by Corollary 4.16, since w and e are on the upper boundary walk of G'_2. Thus, in the embedding $\mathcal{E}_2(G''_2)$ of G''_2 the upper facial walk is the essential 2-face $e'e''$ with minimum v and maximum u, as shown in Figure 4.15. Using again Corollary 4.16 we can also add edges to G'_1 such that lower outer face of $\mathcal{E}_1(G'_1)$ is a 2-face with minimum v and maximum $\max W_1$. By Lemma 4.19 we combine $\mathcal{E}_1(G'_1)$ and $\mathcal{E}_2(G''_2)$ into an embedding containing G. By deleting the added edges, we get a radial embedding of G that is supported by $\mathcal{D}(G)$, which contradicts the choice of G. □

Lemma 4.27. *Suppose that $v, w \in V$ and B is a component of $G \setminus \{v, w\}$ with $v < \min B$ and $\max B < w$. Moreover, there is at least one edge from B to v and at least one from B to w. Then B is essential.*

Proof. We prove this Lemma in a similar way as Lemma 4.25. Therefore, we assume that B is free of essential cycles. Define B' as the subgraph induced by $V(B) \cup \{v, w\}$. We get an x-monotone embedding $\mathcal{E}(B')$ of B' by Lemma 4.22.
There exists a path vPw from v to w in B'. We obtain G' from G by replacing B' with a single curve e' from v to w. Thus, we get the drawing $\mathcal{D}(G')$ of G' by inheriting from $\mathcal{D}(G)$

where $\mathcal{D}(e)$ is obtained from $D(P)$ by suppressing the interior vertices of P. Afterwards, $\mathcal{D}(G')$ is perhaps not radial due to e, but still bounded and independently even. Thus, with Lemma 4.15 we get an independently even radial drawing $D'(G')$. Since G is minimal, we get a radial embedding $\mathcal{E}(G')$ of G'. To obtain a radial embedding of G, we replace e in $\mathcal{E}(G')$ by a "skinny" copy of $\mathcal{E}(B')$ intersecting $\mathcal{E}(G')$ in v and w.
The second part is to show, that the created embedding of G is supported by $D(G)$. Therefore, we assume that there is a essential cycle C in our embedding. By assumption C is not in B. So, either $B' \cap C$ is a path between v and w or C does not intersect B. For the first case, replace the path by the edge e to get an essential cycle C' in the embedding $\mathcal{E}'(G')$, and for the second case C is already an essential cycle in G', so we set $C' = C$. Since the embedding $\mathcal{E}(G')$ of G' is supported by $\mathcal{D}(G')$, there exists an essential cycle C'' in $\mathcal{D}(G')$ with $\ell(C'') \subseteq \ell(C')$. If C'' contains the edge e, replace e by P to get C''' in G. C''' is essential in G, because P can be deformed to e within the cylinder. If otherwise C'' does not contain e, just use C'' as C'''. Thus, we get $\ell(C''') = \ell(C'') \subseteq \ell(C') = \ell(C)$ and the obtained embedding $\mathcal{E}(G)$ of G is supported by $\mathcal{D}(G)$. □

4.3.4 Completing the Proof of Theorem 4.14

To complete the proof, we firstly assume that $D(G)$ is even. But then Theorem 4.2 finishes the proof of part (ii), since we get an embedding of G such that the winding number parity is preserved. Thus, cycles stay essential or non-essential and therefore the embedding is supported by the original drawing.
So, we consider a non-even drawing, that means, there is at least one vertex, which has a pair of edges that cross oddly. If such a pair of edges is in consecutive order in the upper or lower rotation of a vertex v, we can just flip these two edges in a small area around v such that they cross evenly. We repeat this operation until all such consecutive pairs are removed. Afterwards, the only odd pairs that can exists are in a non-consecutive order around a vertex. Assume without loss of generality, that the edges e and f with a common vertex v, lie in the upper rotation at v and are not consecutive, but have a minimum distance in the rotation. Thus, there is at least an edge g between e and f such that g cross both other edges evenly. Actually, the distance in the upper rotation does not matter, only the crossing parity and the order in the rotation. So, we define an unflippable triple.

Definition 4.28. *An* unflippable triple *are three edges a, b and c in that order in the upper rotation at a common vertex v such that the outer pair (a, c) crosses oddly and the other two pairs (a, b) and (b, c) cross evenly.*

The following lemma allows us to rotate the edges of an unflippable triple such that the unflippable triple is preserved.

Lemma 4.29. *If a, b, c is an unflippable triple with common vertex v, we can redraw the ends of a, b and c so their order is b, c, a, and we can redraw the ends at v so their order is c, a, b. In either case, the edges form again an unflippable triple after the redrawing.*

Proof. We can flip a to its right with every edge until it flips with c. So, for all these edges the crossing parity with a changes. In particular the pair (a, b) cross now oddly and (a, c) evenly. The order is then b, c, a, the outer pair (b, a) crosses oddly and the other two pairs (b, c) and (c, a) cross evenly. This is still an unflippable triple, but with cyclically shifted order. By moving c in the original triple to the left and with a similar argumentation, we get that also c, a, b is an unflippable triple. □

The next lemma is a main tool for the rest of the proof. Figure 4.16 shows the setting.

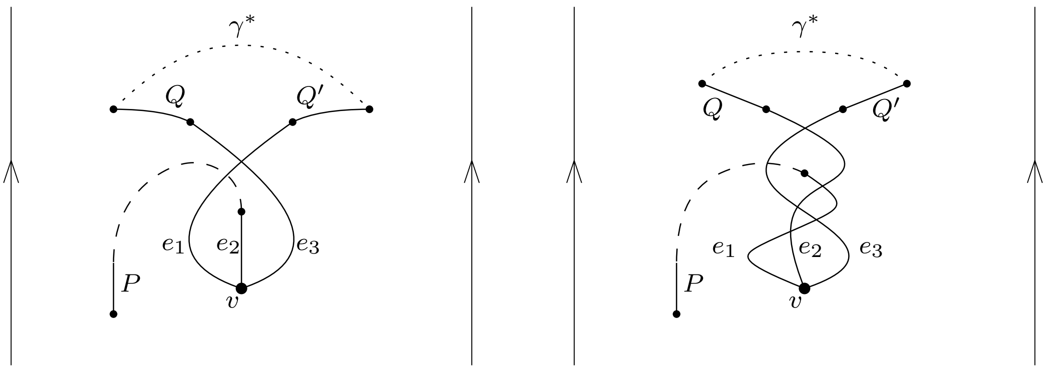


Figure 4.16: Paths P, Q, Q' in a configuration that cannot occur in an independently even radial drawing, if the edges e_1, e_2, e_3 at vertex v form an unflippable triple. (left) P starts at e_2; (right) P starts at e_1.

Lemma 4.30. *Let e_1, e_2, e_3 be three edges in that order in the upper rotation of a common vertex v, such that e_1, e_3 cross oddly and both e_1, e_2 and e_2, e_3 cross evenly. Suppose that P, Q, Q' are paths that begin at v and their first edges are e_1, e_2, e_3, not necessarily respectively in that order, such that $V(P) \cap V(Q) = \{v\} = V(P) \cap V(Q')$ and $v = \min Q = \min Q' > \min P$. (See Figure 4.16) Then it cannot be that both $\max Q > \max P$ and $\max Q' > \max P$.*

Proof. We prove the lemma by contradiction and assume the contrary. We pick the three paths P, Q, Q' to be minimal. That means, that all vertices of the path except for the last one lie in the region $\mathbb{S}^1 \times [\ell(v), \max \ell(P)]$. By adding a simple curve γ^* in the region $\mathbb{S}^1 \times (\max \ell(P), 1)$ joining the endpoints of Q and Q', we get a non-essential curve $\gamma = \gamma^* Q Q'$. Since the end of the path P is in the region $\mathbb{S}^1 \times (0, \ell(v))$, it is in the exterior of γ. Thus, P crosses γ oddly if and only if P starts in the interior of γ, which only occurs if the first edge of P is e_2. If P starts with e_1 or e_3 it crosses γ evenly.
Since γ^* lies in the region $\mathbb{S}^1 \times (\max \ell(P), 1)$, P does not cross γ^*. So, the only possible edges between γ and P that can cross are e_1, e_2 and e_3, because we do not have independent odd edge crossings. If P starts with e_2, it crosses γ evenly, since e_1, e_2 and e_2, e_3 cross evenly. Otherwise if P starts with e_1 or e_3 it crosses γ oddly, since e_1, e_3 have a odd number of crossings and e_1, e_2 and e_2, e_3 an even number. This contradicts the observations from the previous paragraph. □

The remaining proof of Theorem 4.14 splits into two cases, namely whether our considered vertex v with edges e, f and g is $v_1 = \min V(G)$ or not.

Case 1: Assume that $v \neq v_1$.

There has to be a path P through e, f or g ending in the region $\mathbb{S}^1 \times (0, \ell(v))$. If this is not the case, pick a component $G \setminus \{v\}$ containing one of the upper endpoints v_e, v_f, v_g of e, f or g. This component lies in the region $\mathcal{C}(v, 1)$ and is a single vertex by Lemma 4.25 and Lemma 4.26. Thus, the upper endpoints v_e, v_f, v_g are degree-1 vertices in G. Remove the one with the smallest i-coordinate. Without loss of generality let it be v_e together with e. By induction, we get an embedding $\mathcal{E}(G - v_e)$ of $G - v_e$. Afterwards, we embed e along f without any crossings to get a radial embedding for G.
Let P be a minimal path starting at v with e, f or g that minimizes $\max P$ and ends in the region $\mathcal{C}(0, v)$, such that all its vertices except the last one are in the region $\mathcal{C}[v, 1)$. Moreover, let w_P be the maximum vertex of P. Divide the path P at w_P into two subpaths P_1 from v to w_P and P_2 from w_p to the vertex in the region $\mathcal{C}(0, v)$. Let H be the subgraph induced by the vertices $u \in V(G) : v < u < w_p$ between v and w_p. Let H_2 be the component that

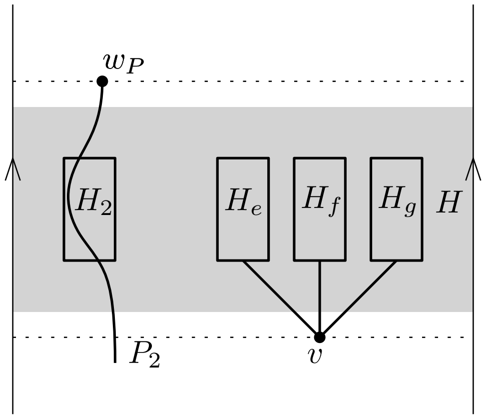


Figure 4.17: The subgraph H (gray region) between v and w_P with its subcomponents.

intersects P_2. If P_2 is only one edge, H_2 is empty. The components H_e, H_f, H_g are the not necessarily distinct components of H incident to e, f, g, respectively. Again, H_e, H_f, H_g can be empty, if the upper endpoint of e, f, g lies above w_p. See Figure 4.17 for an illustration. The subgraph H_2 is disjoint from H_e, H_f, H_g due to the way P is chosen and there is no edge from $H_e \cup H_f \cup H_g$ to the region $\mathcal{C}(0, v)$.

Claim 4.31. *H_e is non-essential and adjacent to a vertex in the region $\mathcal{C}(w_p, 1)$, unless H_e is empty, and likewise for H_f and H_g. (H_e is empty if and only if $\max e \geq w_p$, and likewise for H_f and H_g.)*

Proof of Claim. If H_e contains an essential cycle, this cycle and the path P_2 contradict Lemma 4.17. Moreover, H_e cannot be adjacent to a vertex in the region $\mathcal{C}(0, v)$ due to the choice of P. Assume that H_e is not adjacent to a vertex in the region $\mathcal{C}[w_P, 1)$. Then the only adjacent vertex to H_e is v. But Lemma 4.25 and Lemma 4.26 provide, that H_e is just a single vertex v_e, namely the upper endpoint of e. Removing v_e and e gives us by induction a radial embedding $\mathcal{E}(G - v_e)$ of $G - v_e$, that is supported by $\mathcal{D}(G - v_e)$. If there exists an edge vw' with $w' \geq w_P$ we can embed e along vw' to get an embedding for G. If there is no such edge, P_1 contains at least one vertex in the region $\mathcal{C}(v, w_P)$.
Analogously to H_2, we define H_1 as the component of H that intersects P_1. Thereby, the component H_1 is equal to H_e, H_f or H_g since P starts with e, f or g. Let H_1' be the subgraph induced by $V(H_1) \cup v$. This subgraph is not incedent to an edge intersecting the region $\mathcal{C}(0, v)$ by the choice of P and it does not contain any essential cycle. If H_1' would contain an essential cycle, it would contradict Lemma 4.17, since $\ell(H_1') \subseteq \ell(P_2)$. On top of this, we define H_1'' as the union of H_1' with all incident edges intersecting the region $\mathcal{C}(w_P, 1)$. We can use the boundary of the lower outer face of this subgraph H_1'' in the embedding $\mathcal{E}(H_1'')$, to draw e along it, such that e is bounded. We can redraw this bounded drawing of e with Lemma 4.15 to get an embedding of e without crossings, this is a contradiction.
Hence, if H_e is not empty, it must be adjacent to a vertex different from v in the region $\mathcal{C}[w_P, 1)$, since $\mathcal{C}(0, v)$ is not possible by the choice of P. Similarly H_f and H_g have neighbours in $\mathcal{C}[w_P, 1)$ if they are not empty. □

Claim 4.32. *There exists a cycle C in $G[v, w_P]$ that contains v and exactly two edges in $\{e, f, g\}$.*

Proof of Claim. Without loss of generality we assume, that P goes through e. If this it not the case, we use Lemma 4.29 to reorder the edges at v, such that e is the left edge and the start of P, g is the middle one and f the right one. By Claim 4.31, there is a path from vertex v over f through H_f that ends in $G[w_P, 1)$. Name a minimal such path P_f. Analogously, we define P_g, that starts with the edge g. Let us assume, that neither P_f not P_g intersect with $P_1 \setminus v$. Then both paths do not intersect P. Moreover, both end in $\mathcal{C}(w_P, 1)$, which is a contradiction to Lemma 4.30. Therefore, at least one of the paths P_f or P_g contains a subpath from v to a vertex in P_1 in the region $\mathcal{C}[v, w_P]$. Hence, there is a cycle C in the region $\mathcal{C}[v, w_P]$ containing e, v, f or e, v, g. □

We pick the cycle C in $G[v, w_P]$ such that it contains exactly two of the three edges $\{e, f, g\}$ and minimizes $\max C$. Define $w = \max C \leq w_P$ to be the maximum vertex of C. If the upper endpoint v_e of e is in $G(v, w)$, we define the component of $G(v, w)$ that contains v_e as B_e. The component B'_e is the union of B_e with all incident edges and endpoints, which includes e and v. If otherwise $v_e \geq w$, then $B_e = \emptyset$ and B'_e is just the edge e with its endpoints v and v_e. B_f, B'_f, B_g, B'_g are defined analogously with f and g, respectively. Since $w \leq w_P$, $B_e \subseteq H_e$, $B_f \subseteq H_f$ and $B_g \subseteq H_g$. By the choice of C, neither of the components B_e, B_f, B_g intersect with each other and by P none is adjacent to a vertex in $G(0, v)$.

Claim 4.33. *If C is non-essential, then w is the upper endpoint of the edge in $\{e, f, g\} \setminus E(C)$.*

Proof of Claim. We first treat the case that C contains e and f. Then the edge g starts between e and f around v and since g crosses every edge of C an even number of times, the upper endpoint of g lies in the interior of C or on C. Moreover, every vertex of B_g lies in the interior of C, because C and B_g are disjoint and their edges cross evenly. By the same argumentation, B_g is not connected to any vertex in $G(w, 1)$ and $V(B'_g) \setminus V(B_g) \subseteq \{v, w\}$. Assume B_g has no neighbour in $G[w, 1)$, then $B_g = H_g$. But by Claim 4.31 the component H_g is adjacent to a vertex in $C(w_P, 1)$, which is a contradiction to $w_P \geq w$. If now B_g is adjacent to w, Lemma 4.27 gives us, that B_g is essential. This is also a contradiction since $B_g \subseteq H_g$ and H_g is by Claim 4.31 non-essential. Hence, B_g is empty, the upper endpoint of g lies on C and is therefore by the choice of C the vertex w. So, $g = vw$.
The next case is, if C contains e and g. Again g lies between e and f around v. Thus, f starts in the exterior of C. But since f crosses g evenly and e oddly its endpoint lies in the interior of C or on C. By the same argumentation as before with f and g switched, we get $f = vw$.
The last case, that C contains f and g is similar to the second one, with e starting in the exterior of C. As a result, we get $e = vw$. □

Claim 4.34. *We may assume that C is essential and $w_P = w$, so $B_e = H_e$, $B_f = H_f$ and $B_g = H_g$.*

Proof of Claim. We can split C into two v, w-paths. Assume C is non-essential. Then each of the two paths form with the edge in $\{e, f, g\} \setminus E(C)$ again a cycle with maximum w. But then we can apply Claim 4.33 to each of the new cycles and get that $e = f = g = vw$, which is a contradiction to Lemma 4.23 that G has no multiple edges. Hence, we can choose for C an essential cycle.
If $w_P > w$, then the path P_2 must cross C which contradicts Lemma 4.17. So, $w_P = w$ and $B_e = H_e$, $B_f = H_f$ and $B_g = H_g$. □

Claim 4.35. *If B'_e (or B'_f or B'_g) does not intersect $G(w, 1)$, then B'_e (or B'_f or B'_g) has only one edge, $vw = e$ (or f or g).*

Proof of Claim. Assume that B'_e does not intersect $G(w, 1)$. By the previous Claim 4.34, $w_P = w$ and $B_e = H_e$. Thus, H_e is not adjacent to a vertex in $G(w_P, 1)$.
If now H_e is not empty, H_e is non-essential and has a neighbour in $G[w_p, 1)$ by Claim 4.31. Hence, the only possible neighbour of H_e in $G[w_p, 1)$ is w_P. But then we get a contradiction to Lemma 4.27, since H_e has no neighbour in $G(0, v)$ by the choice of P.
We assume otherwise, that $H_e = \emptyset$. Then B'_e is just the edge e with its endpoints, such that the upper endpoint v_e is in $G[w_p, 1)$. Since B'_e does not intersect $G(w, 1)$, $v_e = w$ and we get $e = vv_e = vw$. The same argumentation works with B'_f and B'_g instead of B'_e which finishes the proof of the claim. □

To end the proof of this case we want to find a contradiction to Lemma 4.23, that there are no multiple edges. For this, we assume without loss of generality, that the cycle C contains e and f. Otherwise, use Lemma 4.29 and renaming.

Claim 4.36. *B'_g intersects $G(w, 1)$.*

Proof of Claim. Assume the contrary. Then by Claim 4.35 the subgraph $B'_g = vw$ and by Claim 4.34 the cycle C is essential. The two v, w-paths of C form with g two new cycles, where at least one must be non-essential, since otherwise C is obtained as a symmetric difference of two essential cycles and therefore non-essential, which is a contradiction. We pick that non-essential cycle and apply Claim 4.33 to this cycle according to the definition of C before Claim 4.33. This gives us, that $e = vw = g$ or $f = vw = g$, what contradicts Lemma 4.23. □

Claim 4.37. *B'_e and B'_f do not intersect $G(w, 1)$.*

Proof of Claim. Assume that B'_e intersect $G(w, 1)$. It works symmetrically with B'_f. Then there exists a path Q in B'_e that starts at v with e and ends in $G(w, 1)$. But there is also a path Q' starting at v with g and ending in $G(w, 1)$ by Claim 4.36. As a third path we pick P' to be the concatenation of P_2 and the v, w-path in $C \cap B'_f$. Then $P' \cap Q = v$ and $P' \cap Q' = v$. Thus, P', Q, Q' apply to Lemma 4.30 through f, e and g and we get a contradiction. □

Hence, by Claim 4.37 and 4.35, $e = vw = f$, which is a contradiction to Lemma 4.23. So, if any edges cross oddly in the upper rotation of a vertex v, we get $v = v_1$. Symmetrically, we get that if any two edges cross oddly in the lower rotation of a vertex v, $v = v_n$.

Case 2: Only pairs of edges incident to v_1 or to v_n may cross oddly.

Firstly, we assume that G does not contain the edge v_1v_n. If G contained v_1v_n, $D(G)$ would be weekly essential, since any essential cycle not containing v_1 or v_n would contradict Lemma 4.17 with the edge v_1v_n. But then Lemma 4.22 gives us a radial embedding.
We modify G to get G' as in Definition 4.20. By Lemma 4.21 we get an even drawing and then by Theorem 4.2 a radial embedding of G'. To obtain a radial embedding of G we redraw the pendent edges of Definition 4.20 and identify the ends to get v_1 and v_n. This must be done carefully to satisfy part (ii) of Theorem 4.14.
We redraw the edges incident to v_1 such that the maximum vertex x of the lower face boundary walk W of G' is also on the outer face boundary walk of G. Assume that W starts and ends at x. Then the remaining vertices of W are ordered by W in a certain way. We order the edges incident to v_1 in exactly that way at v_1, as shown in Figure 4.18. Analogously, we proceed at v_n such that the minimum vertex of the upper boundary walk of G' is on the outer boundary walk of the embedding of G.

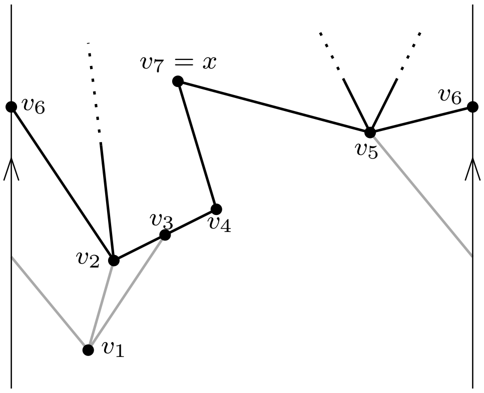


Figure 4.18: The lower part of G' without the pendent edges in black. The edges incident to v_1 are added in grey as they are in the embedding of G such that the maximum vertex $v_7 = x$ of the lower face boundary of G' is on the outer face boundary of G.

Then any essential cycle that is in G but not in G' must contain v_1 or v_n. To fulfill part (*ii*) of Theorem 4.14 the embedding of G' must contain an essential cycle C' with $[\min C', \max C'] \subseteq [\min C, \max C]$. But a lower or upper facial walk of G' contains such a essential cycle. Hence, the proof of Theorem 4.14 is finished.

4.4 Uniform Hanani-Tutte for Radial Drawings

Now we want to state the uniform variant of the theorem.

Theorem 4.38. *If a leveled graph, possible with multiple edges but without loops, has a radial drawing in which every two independent edges cross an even number of times, then it has a radial embedding, such that the rotation system at even vertices is preserved.*

To prove this theorem, we again prove a stronger version from which Theorem 4.38 directly follows. Thereby, we convert the given graph into an ordered graph, apply the stronger version and rebuilt the original graph, analogously to the construction in Section 4.2 to prove Theorem 4.1.

Theorem 4.39. *If an ordered graph G has a radial drawing in which every two independent edges cross an even number of times, then G has a radial embedding, such that the rotation system at even vertices is preserved. Moreover, the new radial embedding is supported by the original drawing.*

Compared to the stronger version of the strong Hanani-Tutte theorem for radial planarity (Theorem 4.14), we left out the part, that G has an x-monotone embedding, if the given drawing of G is weakly essential. This is the case, since Lemma 4.22 does not work in the uniform setting. In the last step of the proof of Lemma 4.22 we combine the new vertices back to one. This merge may change the rotation system at v_1 or v_n, which is not allowed if the respective vertex is even. See Figure 4.19 as an example.

But we can prove a slightly weaker lemma. By adding the condition that G that every essential cycles goes through v_1 and v_1 has an odd crossing.

Lemma 4.40. *Suppose that G has an independently even radial drawing, such that each essential cycle passes through v_1 and v_1 is odd. Then G has an x-monotone embedding.*

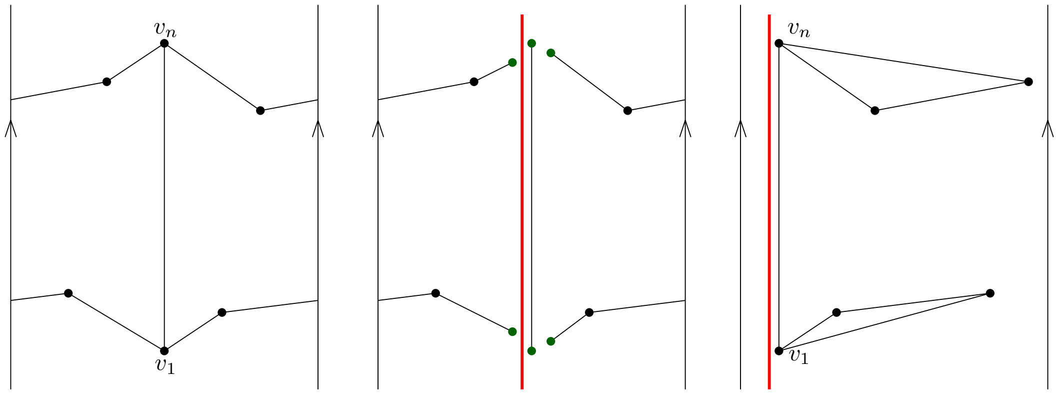


Figure 4.19: A weakly essential graph G with an upper and lower essential cycle. The middle figure shows the split and the last one the combination with a changed rotation system at v_1 and v_n.

Proof. Let $\mathcal{D}(G)$ be the independently even radial drawing, such that each essential cycle passes through v_1 and v_1 is odd. By an adaption of Definition 4.20 and Lemma 4.21 we get a new graph G' such that we only remove v_1 and leaf v_n as it is. Moreover, this gives us an independently even radial drawing $\mathcal{D}(G')$ that contains no essential cycle, since every essential cycle passed through v_1. Now we prove the lemma analogously to Lemma 4.22: We can find a curve e, along which the cylinder can be cut open. By applying the uniform Hanani-Tutte theorem for level planarity [Bö22], we get an x-monotone embedding for G'. To get a drawing for G we combine the added vertices v_i' back to vertex v_1. Since v_1 was odd, we are allowed to change the rotation system, such that we arrange the edges around v_1 in any particular order, so no essential cycles arise. This gives us an embedding for G that is x-monotone and has no essential cycles. □

4.4.1 Components of a Minimal Counterexample

But we are able to prove some requirements for a minimal counter example. Let G be a minimal counterexample with the fewest number of vertices and among them one with the fewest edges. Let $\mathcal{D}(G)$ be an independently even radial drawing of G. Then there is no radial embedding of G that is supported by $\mathcal{D}(G)$ and keeps the rotation at even vertices. We start with the proof, that a minimal counterexample does not contain multiple edges and show afterwards, that the graph must be connected.

Lemma 4.41. *A minimal counterexample does not contain multiple edges.*

Proof. Assume a graph G for a minimal counterexample contains multiple edges e and e' with endpoints u and v. If e and e' cross oddly, u and v are odd vertices and we are allowed to change the rotation system at these vertices. Thus, we proceed as in the proof of Lemma 4.23. Remove e' from G and get by induction a drawing $\mathcal{D}(G-e')$ of $G-e'$. Afterwards draw e' alongside e and get an embedding $\mathcal{E}(G)$ of G. This embedding is supported by the original drawing of G, since for any essential cycle C through e', the cycle $C' = C - e' + e$ is in $G - e'$ and essential with $\ell(C) = \ell(C')$.
Suppose otherwise, that e and e' cross an even number of times. We concatenate e and e' to get a closed curve $\mathcal{C}$. We two-color the complement of this curve such that connected regions get the same color and crossing $\mathcal{C}$ changes the color. Let the colors be red and blue. Our goal is to obtain a drawing of G such that no edge crosses $\mathcal{C}$ an odd number of times. This is the case if every edge of G except of e and e' belong to exactly one of the two colors. Therefore, a vertex $w \in V(G) \setminus \{u, v\}$ is red if it lies completely in a red region

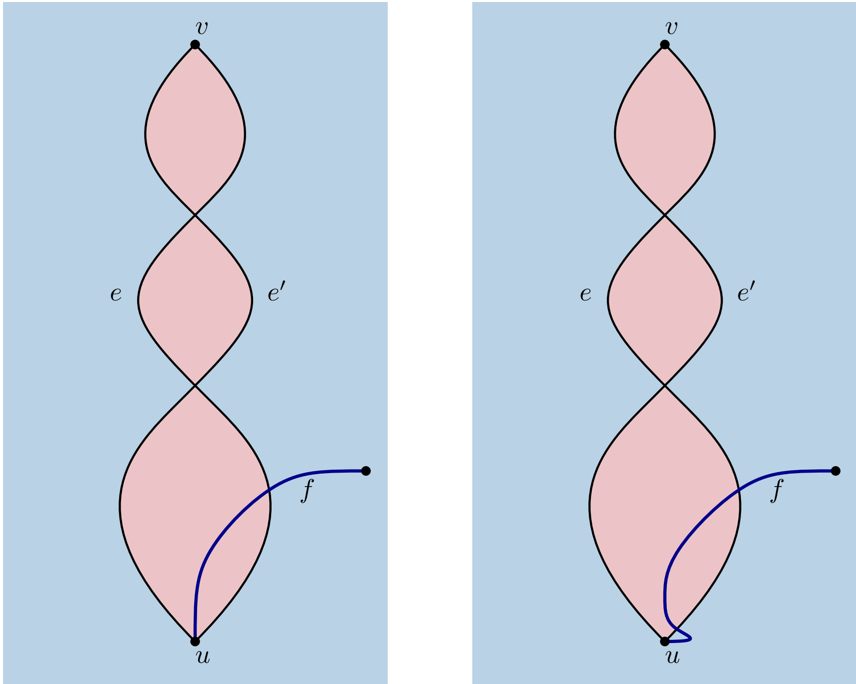


Figure 4.20: Multiple edges e and e' with an edge f that crosses e oddly. In the right figure the local change at u is shown such that f belongs to blue.

and blue if it lies in a blue region. The vertices u and v do not belong to one color, since these vertices lie on the curve $\mathcal{C}$. We now describe how to proceed with the vertex u; the case of v is analogous. If an edge g is adjacent to u, we consider a small area around u. This area is two-colored by $\mathcal{C}$. Then u is red with respect to g, if the area around u, where g is connected to u, is red. Otherwise, u is blue with respect to g, if the entered area is blue. So, the color of u depends on the considered edge g. So, an edge h belongs to red if both endpoints are red and to blue if both endpoints are blue.

We now assume, that there is an edge f whose endpoints have different colors. Then at least one endpoint has to be u or v, because we only get different colored endpoints if f crosses $\mathcal{C}$ and so e and e' an odd number of times. Since the drawing is independently even, f shares an endpoint with e and e', which is u or v. Let without loss of generality u be the common vertex of e, e' and f. We locally change the rotation at u such that only f is moved around u until it connects to u through the other colored region as shown in Figure 4.20. We repeat this operation in any order for all edges that do not belong to one color. Hence, we get a drawing $\mathcal{D}'(G)$ such that each edge of G except of e and e' is either red or blue.

First suppose, that the red or the blue area of the drawing $\mathcal{D}'(G)$ do not contain any edge. Then e and e' lie next to each other in the rotations of u and v. Then, we just remove e' from $\mathcal{D}'(G)$ to get an embedding of $G - e'$ by induction and add e' alongside e such that the rotation at u and v is preserved. This is possible, since e and e' cross evenly by assumption. Then the support-property is fulfilled by the same argumentation as above in the case, where e and e' cross oddly.

Assume now, that e and e' do not form an essential cycle. Define G_{red} to be the subgraph that belongs to red, which should be without loss of generality the interior of the curve C including u and v and G_{blue} to be the blue subgraph including u and v as well as e and e'. Then G_{red} is non-essential, since otherwise e and e' with an essential cycle in G_{red} would contradict Lemma 4.17. So, by induction we get radial embeddings $\mathcal{E}(G_{\text{blue}})$ of G_{blue} and $\mathcal{E}(G_{\text{red}})$ of G_{red}. We add a "skinny" copy of $\mathcal{E}(G_{\text{red}})$ to $\mathcal{E}(G_{\text{blue}})$ between e and e' such that they intersect at u and v. This gives an embedding of G such that the rotation at

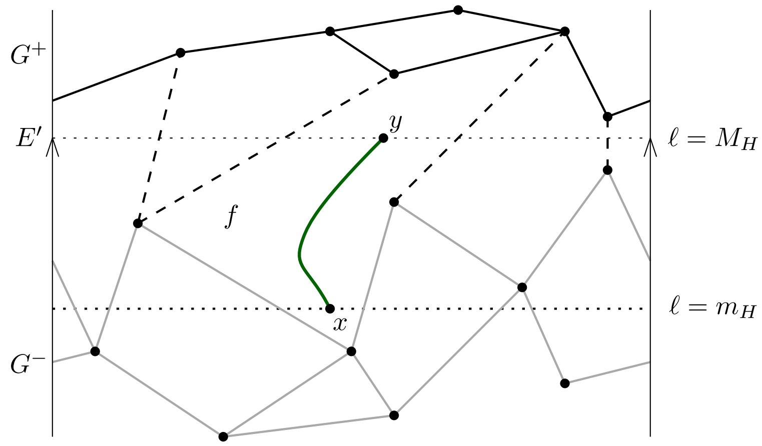


Figure 4.21: The set of edges E' (dashed) and the induced graphs G^- (gray edges) and G^+ (black edges); the green radial curve xy in face f represents the place for $m_H M_H$ and H respectively.

even vertices is preserved.
If e and e' form an essential cycle, this cycle splits the graph into an upper and lower subgraph. Let G_{blue} be e and e' together with the upper subgraph that is without loss of generality blue. Let G_{red} be again e and e' together with the lower subgraph that is in the red region. Then by induction we get a radial embedding $\mathcal{E}(G_{\text{blue}})$ of G_{blue} such that e and e' are the lower face boundary and we get a radial embedding $\mathcal{E}(G_{\text{red}})$ for G_{red} such that e and e' are the upper face boundary. These two embeddings can be merged at u, e, v and e' by Lemma 4.19. This results in a radial embedding of G.
In the previous paragraphs the created embedding of G is always supported by the original drawing $\mathcal{D}(G)$. Hence, a minimal counterexample does not contain multiple edges. □

To prove that a minimal counterexample is connected, we firstly prove that a non-essential component can be added to the embedding of the rest of the graph.

Lemma 4.42. *Let G be a non-connected graph and $\mathcal{D}(G)$ an independently even radial drawing of G. Let H be a non-essential component of G and $G' = G \setminus H$ the remaining graph. Let $\mathcal{E}'(G')$ be an radial embedding of G' such that it is supported by $\mathcal{D}(G')$ and the rotation at even vertices is preserved.*
Then there is an embedding $E(G)$ of G such that $E(G)$ is supported by $\mathcal{D}(G)$ and the rotation at even vertices is preserved.

Proof. We define $m_H = \min V(H)$ and $M_H = \max V(H)$. The goal is to add a crossing-free edge $m_H M_H$ to the embedding $\mathcal{E}'(G')$. By Observation 4.18 this edge is enough such that H can be embedded around the area of $m_H M_H$.
For this, we divide G' into a set of edges E' and two embedded subgraphs G^- and G^+. E' is the set of edges e with $\min e < M_H < \max e$; the set of vertices $\{v \in V(G') : v < M_H\}$ induces G^- and the set of vertices $\{v \in V(G') : v > M_H\}$ induces G^+ as shown in Figure 4.21. By definition, the upper face of G^- contains $\ell = M_H$ and also intersects $\ell = m_H$. If not, the upper boundary of G^- contains an essential cycle between m_H and M_H, which contradicts Theorem 4.39 together with Lemma 4.17 and H. Define a point x with $\ell(x) = m_H$ in the upper boundary of G^- such that x does not intersect with E', but the face f containing x has to intersect $\ell = M_H$. Thus, we can add a curve xy through f from $\ell = m_H$ to $\ell = M_H$, which can be made radial by Corollary 4.16.
To get an embedding of G, we also need an embedding of H. Since H is non-essential, we

get an embedding $E''(H)$ of H that is supported by $\mathcal{D}(H)$ and keeps the rotation at even vertices by the uniform Hanani-Tutte theorem for level-graphs [Bö22]. By Observation 4.18 the edge xy can be replaced by $m_H M_H$ or H. So, we combine the two embeddings $E''(H)$ and $E'(G')$ to an embedding $E(G)$ of G. Thereby no rotation at any even vertex is changed. Moreover, $E(G)$ is supported by $\mathcal{D}(G)$, since the previous operations do not add or remove any essential cycle. □

Lemma 4.43. *G is connected.*

Proof. Assume G is not connected and there is an independently even drawing $\mathcal{D}(G)$. Firstly, suppose there is a non-essential component H of G. Let then $G' = G \setminus H$ be the remaining graph. By induction, we get an embedding $E'(G')$ that is supported by $\mathcal{D}(G')$ and keeps the rotation at even vertices. By Lemma 4.42 we get an embedding of G that is supported by $\mathcal{D}(G)$ and preserves the rotation at even vertices.
The second case is, that every component of G is essential. Let H be the component of G with $\max H = \max G$. We again embed $G' = G \setminus H$ by induction. By the same argumentation as in Lemma 4.42, the upper face of G' must contain $\ell = \max H$ and intersect $\ell = \min H$, since otherwise there exists an essential cycle between m_H and M_H contradicting Theorem 4.39 with H and Lemma 4.17. Thus, the minimum m_U of the upper boundary of G' is below the minimum of H and so G' can be embedded on the cylinder below the curve $\gamma(0, m_U, \pi, \max G')$ by Lemma 4.19. The component H can also be embedded by induction such that the maximum M_L of the lower boundary of H satisfies $\max G' < M_L$. Again by Lemma 4.19 the embedding of H lies strictly above the curve $\gamma(0, \min H, \pi, M_L)$. These two embeddings of G' and H do not intersect, since $m_U < \min H$ and $\max G' < M_L$. Hence, the embeddings of the two subgraphs can be placed above each other with a gap. This is an embedding of G satisfying Theorem 4.39, since all essential cycles lie in a component that was drawn by induction. □

4.4.2 Radial Drawings with Odd Crossings at v_1 or v_n

Given these properties of a minimal counterexample, we want to prove Theorem 4.39, where the only odd crossings appear at v_1 or v_n. This means, there is an unflippable triple (Definition 4.28) at v_1 or v_n. Only such odd crossings are interesting, since if edges, which are consecutive in the rotation of their common vertex, cross oddly, we can just flip these two edges, such that they cross an even number of times.
We start with a lemma, by which we can change an outer face without changing the circular ordering of any vertex.

Lemma 4.44. *Let G be an ordered graph with an even radial drawing $\mathcal{D}(G)$. Let f_i with $i \in \{1, \dots, k\}$ be the faces containing v_1. Then there is a even radial drawing $D_i(G)$ of G such that f_i is an outer face and the circular order of the edges at any vertex stays the same as in $D(G)$.*

Proof. Assume f_i is not the outer face of G, since otherwise we are done. The face f_i has at least two edges at v_1. Let e be the right edge of f_i in the linear upper rotation of v_1. Then starting from e, we have a circular order of the edges around v_1 given by $\mathcal{D}(G)$. We take this ordering and transform it into a linear ordering π_i starting from e. We redraw the incident edges of v_1 in the region $\mathbb{S}^1 \times [\ell(v_1), \ell(v_2)]$ such that they connect to v_1 as defined in π_i without adding crossings or changing the rotation at any other vertex then v_1. After the redrawing the drawing is still even and bounded. By Lemma 4.8 we get the desired even radial drawing $D_i(G)$. An example is shown in Figure 4.22. Hence, the face f_i becomes the outer face and the circular ordering at any vertex remains the same. □

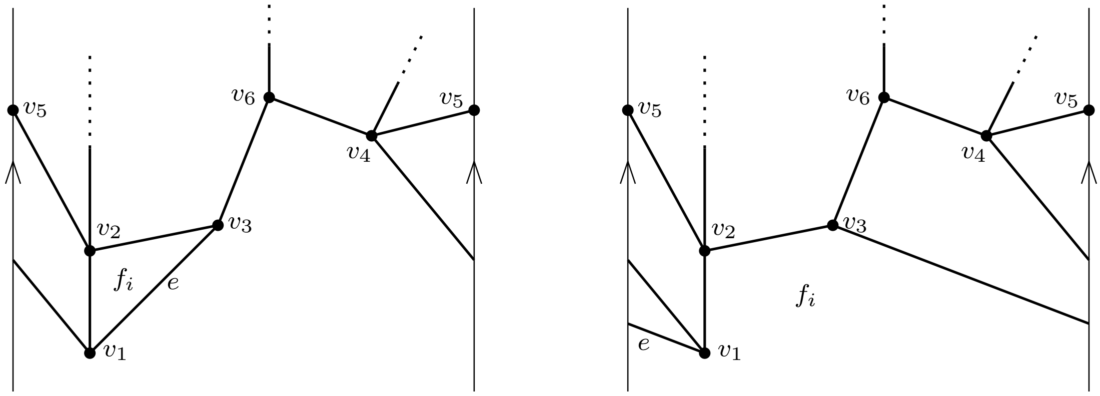


Figure 4.22: Change the drawing such that f_i becomes the outer face, while keeping the circular order at v_1.

Analogously we can redraw the graph at v_n such that every face at v_n can become the outer face.

Corollary 4.45. *Let G be an ordered graph with an even radial drawing $\mathcal{D}(G)$. Let f_i with $i \in \{1, \dots, k\}$ be the faces containing v_n. Then there is a drawing $D_i(G)$ of G such that f_i is the outer face and the circular order of the edges of v_n stays the same as in $D(G)$.*

Now we want to prove the different cases whether there is an odd crossing at v_1 or v_n and whether the edge v_1v_n is present. For this, we introduce a new definition similar to Definition 4.20.

Definition 4.46. *Given an ordered graph G with vertices $v_1 < \dots < v_n$. Define G'_{v_1} to be the ordered graph obtained from G by removing v_1 and replacing the edges incident to v_1. Let w_i with $i \in \{1, \dots, k\}$ be the adjacent vertices of v_1 in G. For G'_{v_1} replace each edge v_1w_i by a new edge v'_iw_i such that the crossings appearing in G are preserved. The vertex v'_i is a new degree-1 endpoint for each edge.*
The graph G'_{v_n} is defined analogously with v_n instead of v_1.

Lemma 4.47. *Let G be an ordered graph with an independently even radial drawing $\mathcal{D}(G)$, such that the only odd crossings appear at v_1. Then there is a radial embedding $\mathcal{E}(G)$ such that $\mathcal{E}(G)$ is supported by $\mathcal{D}(G)$ and the original rotation system at even vertices is preserved.*

Proof. We use Definition 4.46 to obtain G'_{v_1} from G. To remove the odd crossings we apply some radial (e, v)-moves. Let w_i be the upper endpoint of the edge incident to v'_i for $1 \leq i \leq k$. If now v'_iw_i and v'_jw_j cross oddly, we perform a radial (v'_iw_i, v'_j)-move to make these edges cross evenly. We do this for all such pairs: for each $j \in \{1, \dots, k\}$ we perform for each edge v'_iw_i that crosses v'_jw_j oddly a radial (v'_iw_i, v'_j)-move. Afterwards all v'_iw_i, v'_jw_j-pairs cross evenly, and we get an even drawing of G'. By Theorem 4.2, we obtain an embedding $\mathcal{E}'(G'_{v_1})$ of G'_{v_1}, where the rotation of even vertices and the winding number parity of cycles are preserved. To get an embedding of G, we redraw the pendent edges to recreate v_1. It is allowed that the edges at v_1 are reordered, since in $\mathcal{D}(G)$ there were odd crossings at v_1. So, we get a crossing-free radial drawing $\mathcal{D}'(G)$ of G, that may not be supported by $D(G)$.
To achieve that the resulting embedding is supported by $D(G)$, we connect the edges to v_1 such that the maximum vertex x of the lower face boundary walk W of G' is on the

outer face of G. By Lemma 4.44, we pick the face of v_1 containing x and get a drawing such that x is on the outer face of G, as shown in Figure 4.18. This reinsertion of v_1 ensures that the support-property is fulfilled. Since any essential cycle C, that is in G but not in G'_{v_1} has to pass through v_1. Thus, there must be an essential cycle C' in the embedding of G'_{v_1} with $[\min C', \max C'] \subseteq [\min C, \max C]$. But a lower facial walk of G'_{v_1} contains such an essential cycle. Hence, we get an embedding $\mathcal{E}(G)$ of G that is supported by $\mathcal{D}(G)$. Moreover, the rotation at even vertices is preserved, since the only possible changed rotation is at v_1, which is allowed by the odd crossing at v_1. □

Analogously we can embed a graph G whose only odd crossings appear at v_n.

Corollary 4.48. *Let G be an ordered graph with an independently even radial drawing $\mathcal{D}(G)$, such that the only odd crossings appear at v_n. Then there is a radial embedding $\mathcal{E}(G)$ such that $\mathcal{E}(G)$ is supported by $\mathcal{D}(G)$ and the original rotation system at even vertices is preserved.*

In the following we assume for the given drawing $\mathcal{D}(G)$ of our graph G, that there is an odd crossing at v_1 and there is an odd crossing at v_n. All edges with other endpoints cross evenly.

Lemma 4.49. *Let G be an ordered graph with an independently even radial drawing $\mathcal{D}(G)$, such that there are odd crossings at v_1 and v_n, at all other vertices no such odd crossing appear and the edge v_1v_n is not in G. Then there is a radial embedding $\mathcal{E}(G)$ such that $\mathcal{E}(G)$ is supported by $\mathcal{D}(G)$ and the original rotation system at even vertices is preserved.*

Proof. There is an odd crossing at each of the two vertices v_1 and v_n. We modify G as in Definition 4.20 to get G' with pendent edges at both sides of the graph. By Lemma 4.21 we obtain an even drawing and by Theorem 4.2 an embedding $\mathcal{E}'(G')$ of G'. To get an embedding of G, we redraw the pendent edges to recreate v_1 and v_n. It is allowed that the edges at v_1 and v_n are reordered, since in $\mathcal{D}(G)$ were odd crossings at the vertices v_1 and v_n. So, we get a drawing $\mathcal{D}'(G)$ of G. To achieve that the resulting embedding is supported by $D(G)$, we connect the edges to v_1 such that the maximum vertex x of the lower face boundary walk W of G' is on the outer face of G. By Lemma 4.44, we pick the face of v_1 containing x and get a drawing such that x is on the outer face of G, as shown in Figure 4.18. Analogously, we proceed with v_n and Corollary 4.45, such that the minimum vertex y of the upper face boundary walk of G' is on the outer face of G. This reinsertion of v_1 and v_n ensures, that the support-property is fulfilled. Since any essential cycle C, that is in G but not in G' must pass through v_1 or v_n. Thus, there must be an essential cycle C' in the embedding of G' with $[\min C', \max C'] \subseteq [\min C, \max C]$. But a lower or upper facial walk of G' contains such an essential cycle. Hence, we get an embedding $\mathcal{E}(G)$ of G that is supported by $\mathcal{D}(G)$ and preserves the rotation system at even vertices. □

In the previous lemma we assumed, that v_1v_n is not in G. Now we look at the case, when v_1v_n is present. For this, we introduce again a new form of G' from Definition 4.20, namely $G'_{v_1v_n}$.

Definition 4.50. *Given an ordered graph G with vertices $v_1 < \cdots < v_n$ and with the edge v_1v_n. Define $G'_{v_1v_n}$ to be the ordered graph obtained from G by removing v_1 and v_n and replacing the edges to these vertices. Let $v'_1v''_1$ represent v_1v_n in $G'_{v_1v_n}$ such that v'_1 is the lowest vertex and v''_1 the highest vertex in $G'_{v_1v_n}$. Let w_i with $i \in 2, \dots, k$ be the adjacent vertices of v_1 except of v_n in G. For $G'_{v_1v_n}$ replace each edge v_1w_i by a new edge v'_iw_i such*

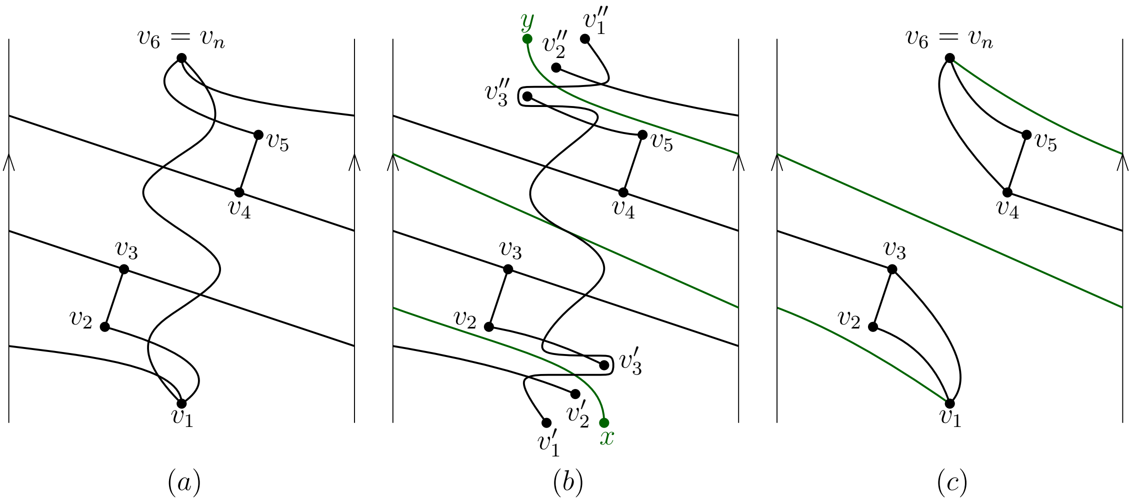


Figure 4.23: Sketch of the steps to get an embedding from a drawing, where the odd crossings appear at v_1 and v_n. Therby, in (a) we have the starting drawing $\mathcal{D}(G)$ and in black in (b) the even radial drawing $\mathcal{D}'(G'_{v_1 v_n})$. In green the new embedding of $v_1'v_1''$ is sketched. The last graphic (c) shows the graph embedded without crossings.

*that the crossings appearing in G are preserved. The vertices v_i' are a new degree-*1 *endpoint for each new edge. Analogously, let w_j' with $j \in 2, \ldots, l$ be the adjacent vertices of v_n except of v_1 in G. For $G'_{v_1 v_n}$ replace each edge $v_n w_j'$ by a new edge $v_j'' w_j'$ such that the crossings appearing in G are preserved. The vertices v_j'' are a new degree-*1 *endpoint for each new edge.*

Lemma 4.51. *Let G be an ordered graph with an independently even radial drawing $\mathcal{D}(G)$, such that there are odd crossings at v_1 and at v_n, at all other vertices no such odd crossing appear and the edge $v_1 v_n$ is in G. Then there is an even radial drawing $\mathcal{D}'(G'_{v_1 v_n})$ of $G'_{v_1 v_n}$ and $\mathcal{D}'(G'_{v_1 v_n})$ is supported by $D(G)$.*

Proof. We use Definition 4.50 to obtain $G'_{v_1 v_n}$ from G. To get an even drawing $D(G'_{v_1 v_n}$ of $G'_{v_1 v_n}$ from $\mathcal{D}(G)$, we redraw the pendent edges of $G'_{v_1 v_n}$. We remove independent odd crossings, which can appear, since there were odd crossing at v_1 and v_n, we do some radial (e, v)-moves. Let w_i be the upper endpoint of an edge incident to v_i' for $1 \leq i \leq k$. If two edges $v_i' w_i$ and $v_j' w_j$ with $i < j$ cross an odd number of times, we perform a radial $(v_i' w_i, v_j')$-move to add one crossing between $v_i' w_i$ and $v_j' w_j$ such that they cross evenly. We apply this procedure to all pairs. So, for each $j \in 1, \ldots, k$ we perform for each edge $v_i' w_i$ that crosses $v_j' w_j$ oddly a radial $(v_i' w_i, v_j')$-move. Afterwards all $v_i' w_i, v_j' w_j$-pairs cross evenly. Analogously, we perform on the other side with the edges $v_i'' u_i$. Then in the drawing $\mathcal{D}'(G'_{v_1 v_n})$ of $G'_{v_1 v_n}$ all edges are even, since we handled the pendent edges and the other edges are already even. That $\mathcal{D}(G)$ supports $\mathcal{D}'(G'_{v_1 v_n})$ follows, since every essential cycle in $\mathcal{D}'(G'_{v_1 v_n})$ is also in $\mathcal{D}(G)$. □

Lemma 4.52. *Let G be an ordered graph with an independently even radial drawing $\mathcal{D}(G)$, such that there are odd crossings at v_1 and v_n, at all other vertices no such odd crossing appear and the edge $v_1 v_n$ is in G. Then there is a radial embedding $\mathcal{E}(G)$ such that $\mathcal{E}(G)$ is supported by $\mathcal{D}(G)$ and the rotation system at even vertices is preserved.*

Proof. We use Definition 4.50 to obtain $G'_{v_1 v_n}$ from G. By Lemma 4.51, we get an even radial drawing $\mathcal{D}'(G'_{v_1 v_n})$ of $G'_{v_1 v_n}$. The graph $G'_{v_1 v_n}$ has two components, namely $v_1' v_1''$

and $G''_{v_1v_n} = G'_{v_1v_n} \setminus v'_1v''_1$. See Figure 4.23 ($a$) and ($b$) for an example. Since $\mathcal{D}'(G''_{v_1v_n})$ is even, we use Theorem 4.2 to get an embedding $\mathcal{E}'(G''_{v_1v_n})$ of $G''_{v_1v_n}$ such that the rotation at even vertices and the winding number parity for cycles is preserved. With Lemma 4.42 we get an embedding $\mathcal{E}'(G'_{v_1v_n})$ for the whole graph $G'_{v_1v_n}$. In Figure 4.23 (b) the green edge sketches, how to redraw $v'_1v''_1$, which corresponds to v_1v_n in G.
To get an embedding of G, we redraw the pendent edges to recreate v_1 and v_n. In part (c) of the figure the embedding of G is sketched. It is allowed that the edges at v_1 and v_n are reordered, since in $\mathcal{D}(G)$ were odd crossings at the vertices v_1 and v_n. So, we get a drawing $\mathcal{D}'(G)$ of G.
To achieve that the resulting embedding is supported by $D(G)$, we connect the edges to v_1 and v_n such that v_1v_n is on the outer face of G and every possible essential cycle contains v_1v_n. This is possible, since we can reorder the edges at v_1 and v_n, and we only get an essential cycle in $\mathcal{E}(G)$, if there was already one in $\mathcal{D}(G)$. Since v_1 and v_n are the extreme vertices, the property that there must be an essential cycle C' in $\mathcal{D}(G)$ for every essential cycle in $\mathcal{E}(G)$ with $[\min C', \max C'] \subseteq [\min C, \max C]$ is fulfilled. □

The entirety of the previous lemmata shows, that we can prove Theorem 4.38, if the only odd crossings appear at v_1 or v_n.

Corollary 4.53. *If an ordered graph G has a radial drawing $\mathcal{D}(G)$ in which every two independent edges cross an even number of times and the only odd crossings appear at v_1 or v_n, then G has a radial embedding, such that the rotation system at even vertices is preserved. Moreover, the new radial embedding is supported by the original drawing.*

Proof. Assume that in $\mathcal{D}(G)$ the only odd crossings are either at v_1 or at v_n. Then Lemma 4.47 and Corollary 4.48 provide the intended result. If we have odd crossings at both vertices v_1 and v_n, we distinguish, whether v_1v_n is in G or not. If v_1v_n is in G we use Lemma 4.52, and if v_1v_n is not in G we use Lemma 4.49. □

We have shown, that Theorem 4.39 holds, if the odd crossings only appear at v_1 or v_n. This matches the end of the proof of Theorem 4.14. To complete the proof of Theorem 4.39, one has to extend the properties of a minimal counterexample about cut-vertices and the corresponding components, see Lemma 4.25 - 4.27. Then the goal is to prove analogously to Case 1 that the odd crossings can only appear at v_1 and v_n.
A possible starting point is, to assume that the given graph is 3-connected. Then G does not contain any cut-vertices or split-pairs.

Conjecture 4.54. *If an ordered 3-connected graph G has a radial drawing in which every two independent edges cross an even number of times, then G has a radial embedding, such that the rotation system at even vertices is preserved. Moreover, the new radial embedding is supported by the original drawing.*

Here, the three Lemmata 4.25 - 4.27 are no longer interesting, since there is no possible cut-vertex v or split-pair (v, w) to create the necessary component B. Moreover, any two planar embeddings of a 3-connected graph are equivalent [Die17, Whitney 1933]. This means there is a unique solution except for mirroring. So, perhaps one can adapt the proof of Theorem 4.14 at Case 1 to prove Conjecture 4.54.
Building on this, assume that the given graph is 2-connected. Thus, there are still no cut-vertices, but now a split-pair can exist.

Conjecture 4.55. *If an ordered* 2*-connected graph* G *has a radial drawing in which every two independent edges cross an even number of times, then* G *has a radial embedding, such that the rotation system at even vertices is preserved. Moreover, the new radial embedding is supported by the original drawing.*

In this setting Lemma 4.25 and 4.26 are still not interesting, since there are no cut-vertices. But the property described in Lemma 4.27 probably can be proven for a minimal counterexample to Conjecture 4.55, because split-pairs can be present. Again, one can then perhaps adapt the proof of Theorem 4.14 at Case 1 to prove Conjecture 4.55.
As a last step one can then probably use the proofs found for the two Conjectures 4.54 and 4.55 together with the results in this chapter (Lemma 4.41, Lemma 4.43 and Corollary 4.53) to finish the proof of Theorem 4.38.

5. Conclusion

We considered different versions of Hanani-Tutte theorems and wanted to extend the proof of the strong Hanani-Tutte theorem to a proof for the uniform version. We started on the projective plane and have stated the proof for the weak and strong Hanani-Tutte theorem. Afterwards, we tried to extend the proof of the strong version to prove that the uniform Hanani-Tutte conjecture for the projective plane also holds. But here a main problem occurred, namely that the representation of projective Hanani-Tutte drawings in the proof of the strong Hanani-Tutte theorem does not work in the uniform setting. So, we were not able to extend the proof of the strong version. To deal with the problem, there are multiple possible ways to continue: We can find another suitable representation for projective Hanani-Tutte drawings and can adapt the proof of the strong version appropriately to get a proof for the uniform Hanani-Tutte theorem on the projective plane. Another approach for a proof is to use the minimal forbidden minors for the projective plane, as Pelsmajer, Schaefer and Stasi [PSS09] did for the strong version, but here it could be difficult to fulfill the property that the rotation at even vertices is preserved. Hence, one probably has to find a complete new proof for Conjecture 3.30. On the other hand, it is also possible, that the uniform Hanani-Tutte conjecture for the projective plane is not true. Then one can probably find a counterexample.

In the second part of the thesis, we considered the Hanani-Tutte theorems for radial planarity. We presented the weak Hanani-Tutte theorem and showed a proof of the strong version of the theorem. Based on this proof, we tried to prove the uniform Hanani-Tutte theorem. We were able to show, that a minimal counterexample has to be connected and does not contain multiple edges. Moreover, we proved that the uniform Hanani-Tutte theorem is true, if odd crossings appear only at v_1 and v_n.
To complete the proof of the uniform Hanani-Tutte theorem for radial planarity, one has to extend the properties of a minimal counterexample. For this, one can first consider 3-connected and 2-connected graphs. Afterwards, the goal is to show the properties of a minimal counterexample about cut-vertices and the corresponding components. Then one has to show analogously to the proof of the strong version that the odd crossings can only appear at v_1 and v_n. This would complete the proof of the uniform Hanani-Tutte theorem for radial planarity.

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