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On Pointwise Hölder Continuity of the Joint Spectral Radius

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Abstract: We show that the joint spectral radius is pointwise Hölder continuous. This extends results from classical perturbation theory of single matrices to the setting of compact sets of matrices. The results are new for reducible matrix sets, as Lipschitz continuity of the joint spectral radius restricted to irreducible matrix sets was already shown by the second author in Wirth (2002).

Keywords: control of switched systems, linear systems, joint spectral radius, switched systems, perturbation theory, Hölder continuity

1. INTRODUCTION

The joint spectral radius of a given set of (real or complex) square matrices measures the worst case growth rate among all possible products of matrices taken from the set. The quantity is of interest in the context of discrete-time switched systems as it characterizes exponential stability under arbitrary switching. The quantity has also been of interest in further applications in wavelet theory, coding, graph theory to name a few. It has been known for a long time that the joint spectral radius is a continuous function of the data. In this paper we study further regularity properties.

The interest in continuity properties of the joint spectral radius stems from numerical considerations on the one hand, as error estimates for numerical schemes typically rely on some sort of continuity property. Also in the context of robustness analysis of dynamical systems, regularity properties provide a tool for the assessment of sensitivity with respect to the system data.

The simplest incarnation of the joint spectral radius is the spectral radius $\rho(M)$ of a single matrix $M \in \mathbb{C}^{d\times d}$. The growth rate of the powers of a square matrix is determined by the spectral radius of the matrix, i.e. the largest modulus of an eigenvalue of M. Studying the dependence of this quantity on the entries of a matrix is a central topic in perturbation theory. It is easy to see that the spectral radius of a matrix does not depend in a Lipschitz continuous manner on the entries of the matrix by considering the perturbation of the lower left entry of an upper triangular Jordan block. Nevertheless, the following classical theorem of Elsner shows that in dimension d the spectral radius is locally Hölder continuous with exponent $\frac{1}{d}$, see (Stewart and Sun, 1990, Theorem IV.1.3). In the following $\|\cdot\|_2$ denotes the spectral norm, i.e. the operator norm induced by the Euclidean norm.

Theorem 1. (Elsner). For $A, B \in \mathbb{C}^{d\times d}$ we have

$$|\rho(A) - \rho(B)| \leq (\|A\|_2 + \|B\|_2)^{(d-1)/d} \, \|A-B\|_2^{1/d}.$$

Corollary 2. The spectral radius $\rho : \mathbb{C}^{d\times d} \to [0,\infty)$ is locally $\frac{1}{d}$-Hölder continuous.

Our aim in this work is to prove a similar result for compact sets of non-commuting matrices. The maximal growth rate of products in this case is controlled by the *joint spectral radius* introduced in Rota and Strang (1960). For the last 20 years this quantity has received considerable attention and a extensive body of results has been obtained in this area, see the monograph Jungers (2009), the milestone papers Berger and Wang (1992); Gurvits (1995); Lagarias and Wang (1995); Bousch and Mairesse (2002); Hare et al. (2011), the surveys Margaliot (2006); Shorten et al. (2007) and references therein.

In this setting we obtain the following analogue of Theorem 1. Let $\mathcal{H}(d)$ denote the space of compact sets of complex $d\times d$ matrices endowed with the Hausdorff metric, which we denote by d_H.

Theorem 3. The joint spectral radius $\rho : \mathcal{H}(d) \to [0,\infty)$ is pointwise $\frac{1}{d^2+d}$-Hölder-continuous.

After we unwrap the definitions, the previous theorem says that for every $\mathcal{M} \in \mathcal{H}(d)$ there are constants $\eta > 0$ and $C > 0$ such that for every $\mathcal{N} \in \mathcal{H}(d)$ with $d_H(\mathcal{M},\mathcal{N}) \leq \eta$ we have $|\rho(\mathcal{N}) - \rho(\mathcal{M})| \leq C d_H(\mathcal{M},\mathcal{N})^{1/(d^2+d)}$. In general, it is necessary to distinguish between local and pointwise Hölder continuity. As C can depend on $\mathcal{M}$ our result claims a weaker statement than local Hölder continuity of ρ. We stress that while the spectral radius of matrices is locally Hölder continuous (i.e. the constant C may be chosen uniformly on suitable neighborhoods) we only obtain pointwise Hölder continuity for the joint spectral radius.

The exponent $\frac{1}{d^2+d}$ is probably not optimal and is a consequence of our methods. If we restrict our attention to finite sets of matrices, we can improve the Hölder exponent to the almost optimal $\frac{1}{d+\varepsilon}$ for arbitrary $\varepsilon > 0$.

Continuity of the joint spectral radius was shown in Barabanov (1988). Local Lipschitz continuity on the set of irreducible compact subsets of $\mathbb{C}^{d\times d}$ was obtained in Wirth (2002). A different proof which also yields concrete estimates for the Lipschitz constants is due to Kozyakin, Kozyakin (2010). The property was extended to the set of positive inclusions with a strongly connected system graph in Mason and Wirth (2014).

Our proof of pointwise Hölder continuity is split into proving a lower and an upper bound for the joint spectral radius of a perturbed matrix set. We use very different methods for these two types of bounds. The upper bound is obtained using ε-inflation and extremal norms. The lower bound on the other hand uses quantitative Berger-Wang estimates and classical perturbation theory in the form of Elsner's theorem.

2. DEFINITIONS AND NOTATION

Let $\|\cdot\|$ be an arbitrary fixed norm on $\mathbb{C}^d$. For a matrix $A \in \mathbb{C}^{d\times d}$ we denote by $\|A\|$ the corresponding operator norm of A. The joint spectral radius of $\mathcal{M} \in \mathcal{H}(d)$ is defined as

$$\rho(\mathcal{M}) := \lim_{n\to\infty} \sup\{\|A_n \dots A_1\|^{1/n} \ ; \ A_1, \dots, A_n \in \mathcal{M}\}.$$

The equivalence of all norms on finite dimensional vector spaces shows that this definition is independent of the choice of $\|\cdot\|$. Rota and Strang (1960) gave another characterization of the joint spectral radius in terms of operator norms. For a given norm $|||\cdot|||$ on $\mathbb{C}^{d\times d}$ define

$$|||\mathcal{M}||| := \max\{|||A|||; A \in \mathcal{M}\}.$$

Then we have

$$\rho(\mathcal{M}) = \inf\{|||\mathcal{M}||| \ ; \ |||\cdot||| \text{ is an operator norm}\}. \quad (1)$$

A norm $\|\cdot\|$ on $\mathbb{C}^d$ is called *extremal* for $\mathcal{M}$, if for the corresponding operator norm we have $\rho(\mathcal{M}) = \|\mathcal{M}\|$.

Recall that $\mathcal{M} \in \mathcal{H}(d)$ is called *reducible*, if there is a nontrivial subspace $X \subsetneq \mathbb{C}^d$ such that X is A-invariant for all $A \in \mathcal{M}$. For a reducible $\mathcal{M}$ there is a maximal flag $\{0\} = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_m = \mathbb{C}^d$ such that every subspace F_j, $j = 0, \dots, m$, is A-invariant for all $A \in \mathcal{M}$. We will call the length m of this maximal flag the *reducibility index* of $\mathcal{M}$. Note that with this convention a reducibility index of 1 means that the system is in fact irreducible, i.e. no nontrivial joint invariant subspace exists.

An extremal norm $\|\cdot\|$ is called a *Barabanov norm*, if in addition for every $x \in \mathbb{C}^d$ there exists an $A \in \mathcal{M}$ such that

$$\|Ax\| = \rho(\mathcal{M}) \|x\| .$$

A sufficient condition for the existence of Barabanov norms is that the set $\mathcal{M}$ is irreducible, Barabanov (1988); Wirth (2002).

3. THE UPPER BOUND USING ε-INFLATION

In this section we study the behavior of the joint spectral radius under the addition of a ball of radius ε to a matrix set. This will give us an upper bound on the joint spectral radius of a perturbed matrix set. For a norm $\|\cdot\|$ on $\mathbb{C}^d$ let $\mathcal{B}^{\|\cdot\|} = \{A \in \mathbb{C}^{d\times d} \ ; \ \|A\| \le 1\}$ be the unit ball of the corresponding operator norm. By ε-inflation we mean the study of the increasing set-valued map

$$\varepsilon \mapsto \mathcal{M} + \varepsilon\mathcal{B}^{\|\cdot\|},$$

where $\mathcal{M} \in \mathcal{H}(d)$ and the addition is in the sense of Minkowski. For this case, the following result shows Hölder-continuity of the joint spectral radius as a function of ε.

Proposition 4. Let $\|\cdot\|$ be a norm on $\mathbb{C}^d$ and let $\mathcal{M} \in \mathcal{H}(d)$ have index of reducibility m. Define

$$\begin{aligned}\mathcal{M}_\varepsilon^{\|\cdot\|} &:= \mathcal{M} + \varepsilon\mathcal{B}^{\|\cdot\|},\\ &:= \{A + \varepsilon B \ ; \ A \in \mathcal{M}, B \in \mathcal{B}^{\|\cdot\|}\}, \varepsilon > 0.\end{aligned}$$

Then

$$r : \varepsilon \mapsto \rho\left(\mathcal{M}_\varepsilon^{\|\cdot\|}\right)$$

is increasing. In addition, it is Hölder continuous at 0 with exponent $1/m$. In particular, for any $\eta > 0$ there exists a constant C_η such that

$$r(\varepsilon) - r(0) \le C_\eta \varepsilon^{1/m}, \quad \varepsilon \in [0, \eta]. \quad (2)$$

Proof. It is clear that r is increasing. For reducibility index $m = 1$ we are in the irreducible case and the result follows from Wirth (2002).

Let $\mathcal{M}$ have reducibility index $m \ge 2$. It is sufficient to prove the result for one specific norm.

By the equivalence of norms, for given norms v_1 and v_2 there is a constant D such that $\varepsilon\mathcal{B}_V^{v_2} \subseteq D\varepsilon\mathcal{B}_V^{v_1}$ and so $\rho(\mathcal{M}_\varepsilon^{v_2}) \le \rho(\mathcal{M}_{D\varepsilon}^{v_1})$. Thus if the result is known for v_1 with a constant C_η on the interval η, then we have on $[0, \eta/D]$ that

$$\rho(\mathcal{M}_\varepsilon^{v_2}) - \rho(\mathcal{M}) \le \rho(\mathcal{M}_{D\varepsilon}^{v_1}) - \rho(\mathcal{M}) \le C_\eta D^{1/m} \varepsilon^{1/m}. \quad (3)$$

We thus begin by fixing a suitable norm. Let $(F_0, \dots, F_m)$ be a flag corresponding to the reducibility index of $\mathcal{M}$ and choose pairwise orthogonal spaces $X_1, \dots, X_m$ such that

$$F_{i-1} \oplus X_i = F_i, \quad i = 1, \dots, m. \quad (4)$$

Let $\pi_i : \mathbb{C}^d \to X_i$ be the orthogonal projection and $\imath_i : X_i \to \mathbb{C}^d$ the canonical injection. We define

$$\mathcal{M}_{ij} := \{\pi_i A \imath_j \ ; \ A \in \mathcal{M}\}, \quad 1 \le i, j \le m. \quad (5)$$

The sets $\mathcal{M}_{ii}$ of the restrictions of $\pi_i A$ to X_i, $A \in \mathcal{M}$, are irreducible or equal to $\{0\}$ and so, Barabanov (1988); Wirth (2002), we may choose Barabanov norms v_i for $\mathcal{M}_{ii}$, $i = 1, \dots, m$. In particular, we have (see also (Berger and Wang, 1992, Lemma 2))

$$\rho(\mathcal{M}) = \max_{i=1,\dots,m} \rho(\mathcal{M}_{ii}) = \max_{i=1,\dots,m} v_i(\mathcal{M}_{ii}). \quad (6)$$

Define a norm v on $\mathbb{C}^d$ by

$$v(x) = \|(v_1(\pi_1(x)), \dots, v_m(\pi_m(x)))\|_2 . \quad (7)$$

A calculation shows that the operator norm induced by v satisfies

$$v(A) \le \left\| \begin{bmatrix} v_{11}(\pi_1 A \imath_1) & \dots & v_{1m}(\pi_1 A \imath_m) \\ \vdots & & \vdots \\ v_{m1}(\pi_m A \imath_1) & \dots & v_{mm}(\pi_m A \imath_m) \end{bmatrix} \right\|_2 \quad (8)$$

for any $A \in \mathbb{C}^{d\times d}$, where we denote by v_{ij} the induced operator norm from (X_j, v_j) to (X_i, v_i).

By similarity scaling we may assume without loss of generality that

$$v_{ij}(\mathcal{M}_{ij}) \le 1, \quad 1 \le i < j \le m.$$

On the other hand of course $\mathcal{M}_{ij} = \{0\}$ for $i > j$.

It suffices to show that there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$ we have

$$\rho(\mathcal{M}_\varepsilon^v) - \rho(\mathcal{M}) \leq C\varepsilon^{1/m}.$$

The full claim then follows by an easy calculation.

Using the properties of v, for $\varepsilon > 0$ we then have for the blocks of the matrices $A \in \mathcal{M}_\varepsilon^v$ that

$$v_{ij}(\mathcal{M}_{\varepsilon,ij}) \leq \begin{cases} (1+\varepsilon)\rho(\mathcal{M}_{ii}) & \text{for } i = j, \\ (1+\varepsilon) & \text{for } i < j, \\ \varepsilon & \text{for } i > j. \end{cases} \quad (9)$$

As the joint spectral radius is invariant under similarity transformation, we can now rescale via a diagonal transformation of the form

$$T_\varepsilon = \operatorname{diag}(I_1, \delta I_2, \dots, \delta^{m-1} I_m),$$

where I_j is the identity matrix of dimension $\dim X_j$, $j = 1, \dots, m$ and

$$\delta = \sqrt[m]{\varepsilon}. \quad (10)$$

A calculation shows for any matrix in $A \in T_\varepsilon^{-1}\mathcal{M}_\varepsilon^v T_\varepsilon$ that

$$v(A) \leq \left\| \begin{bmatrix} (1+\varepsilon)\rho(\mathcal{M}_{11}) & \dots & (1+\varepsilon)\varepsilon^{(m-1)/m} \\ \varepsilon^{(m-1)/m} & & \vdots \\ \vdots & & (1+\varepsilon)\varepsilon^{1/m} \\ \varepsilon^{1/m} & \dots & (1+\varepsilon)\rho(\mathcal{M}_{mm}) \end{bmatrix} \right\|_2 . \quad (11)$$

Denoting the matrix on the right by $Q(\varepsilon) = (q_{ij}(\varepsilon))_{i,j=1}^m$, we see that the diagonal entries of $Q(\varepsilon)$ are of the form

$$q_{jj}(\varepsilon) = (1+\varepsilon)\rho(\mathcal{M}_{jj}), \quad j = 1, \dots, m,$$

while we have for the off-diagonal entries that

$$0 \leq q_{ij}(\varepsilon) \leq 2\varepsilon^{1/m}, \quad i \neq j,$$

for $\varepsilon \in [0, 1]$. We also note that

$$\rho(\mathcal{M}) = \max_{i=1,\dots,m} \rho(\mathcal{M}_{ii}) = \|Q(0)\|_2 .$$

Consequently, we have

$$\begin{aligned} \rho(\mathcal{M}_\varepsilon^v) - \rho(\mathcal{M}) &= \rho(T_\varepsilon^{-1}\mathcal{M}_\varepsilon^v T_\varepsilon) - \rho(\mathcal{M}) \\ &\leq v(T_\varepsilon^{-1}\mathcal{M}_\varepsilon^v T_\varepsilon) - \rho(\mathcal{M}) \\ &\leq \|Q(\varepsilon)\|_2 - \|Q(0)\|_2 \\ &\leq \|Q(\varepsilon) - Q(0)\|_2 \leq C\varepsilon^{1/m} \end{aligned}$$

for all $0 \leq \varepsilon \leq 1$ and a suitable positive constant C. □

For the particular case of ε-inflation the previous result allows for the following stronger statement in which we even obtain a local Hölder continuity result.

Theorem 5. Let $\mathcal{M} \in \mathcal{H}(d)$ be of reducibility index $m \geq 1$. The map $r : [0, \infty) \to \mathbb{R}$

$$r(\varepsilon) := \rho(\mathcal{M}_\varepsilon^v)$$

is locally $\frac{1}{m+1}$-Hölder continuous, if $m \geq 2$, and locally Lipschitz continuous if $m = 1$.

For a proof we refer the reader to Epperlein and Wirth (2023).

Finally, we have the following bound which provides the upper estimate in the pointwise Hölder estimate.

Theorem 6. Let $\mathcal{M} \in \mathcal{H}(d)$. For every $\eta > 0$ there is a constant C_η such that $\rho(\mathcal{N}) \leq \rho(\mathcal{M}) + C_\eta d_H(\mathcal{M}, \mathcal{N})^{1/d}$ for all $\mathcal{N} \in \mathcal{H}(d)$ with $d_H(\mathcal{M}, \mathcal{N}) \leq \eta$.

Proof. The result follows directly from the previous Proposition 4 and the fact that $\mathcal{N} \subseteq \mathcal{M}_\varepsilon^{\|\cdot\|}$ for $\varepsilon = d_H(\mathcal{M}, \mathcal{N})$.

Note that with the previous result we have already obtained pointwise Hölder continuity in all $\mathcal{M} \in \mathcal{H}(d)$ with the property that $\rho(\mathcal{M}) = 0$.

4. THE LOWER BOUND USING BERGER-WANG ESTIMATES

To get a lower bound on the joint spectral radius of a perturbed matrix set, we use approximations of the joint spectral radius from below by the spectral radius of a sufficiently long product. An application of Elsner's theorem, Theorem 1, to this product then yields the result.

For the required estimates we need to collect a few results from the literature.

The statement of the following proposition is basically contained in the introduction of Morris (2010), but we need slightly more explicit estimates.

Proposition 7. There is a constant Λ_d depending only on the dimension d such that for all bounded sets of matrices $\mathcal{M}$ in $\mathbb{C}^{d\times d}$ we have

$$\max_{1\leq k\leq n} \sup_{A_1,\dots,A_k\in\mathcal{M}} \rho(A_k \dots A_1)^{1/k} \geq \rho(\mathcal{M}) \left(1 - \frac{\Lambda_d}{n}\right).$$

The following exponential-polynomial growth bound can be found in Varney and Morris (2022) and is also an immediate consequence of (Guglielmi and Zennaro, 2001, Theorem 3.1). The idea is to use the fact (due to Barabanov) that a family of matrices with joint spectral radius equal to 1 and unbounded growth must be reducible.

Lemma 8. Let $\mathcal{M} \in \mathcal{H}(d)$ be a compact set of matrices with $\rho(\mathcal{M}) > 0$. There is a constant $\Theta > 0$ depending on $\mathcal{M}$ such that

$$\|A_k \dots A_1\| \leq \Theta k^{d-1} \rho(\mathcal{M})^k \quad (12)$$

for all $k \in \mathbb{N}$, $A_i \in \mathcal{M}$, $i = 1, \dots, k$.

The next lemma is obtained by rather elementary norm estimates.

Lemma 9. Let $\mathcal{M} \in \mathcal{H}(d)$ be a compact set of matrices with $\rho(\mathcal{M}) = 1$. Let Θ be the constant from Lemma 8 for the matrix set $\mathcal{M}$. Then

$$\|(A_k + \varepsilon B_k) \cdots (A_1 + \varepsilon B_1) - A_k \cdots A_1\| \leq 2\Theta^2 \varepsilon k^{2d-1}$$

for all $k \geq 1$, $\varepsilon < \frac{1}{\Theta} k^{-d}$, $A_i \in \mathcal{M}$, $\|B_i\| \leq 1$, $i = 1, \dots, k$.

With this preparation in hand, we are ready to prove the central result in this section.

Theorem 10. Let $\mathcal{M} \in \mathcal{H}(d)$. For every $\eta > 0$ there is a constant C_η such that $\rho(\mathcal{N}) \geq \rho(\mathcal{M}) - C_\eta d_H(\mathcal{M}, \mathcal{N})^{1/(d^2+d)}$ for all $\mathcal{N} \in \mathcal{H}(d)$ with $d_H(\mathcal{M}, \mathcal{N}) \leq \eta$.

Proof. (Sketch) It is enough to show the theorem for some $\eta \leq 1$. For $\rho(\mathcal{M}) = 0$ the conclusion of the theorem is trivially satisfied. By rescaling we may assume $\rho(\mathcal{M}) = 1$.

We set $\eta := n_0^{-(d^2+d)}$ for a sufficiently large positive integer n_0, which will be specified later. Let $\mathcal{N}$ be a compact set of matrices with $d_H(\mathcal{M}, \mathcal{N}) =: \varepsilon \leq \eta$.

Next pick a positive integer $n \geq n_0$ such that

$$\varepsilon \in \left(\frac{1}{(2n)^{d^2+d}}, \frac{1}{n^{d^2+d}} \right]. \quad (13)$$

By Proposition 7 we can find $k \leq n$ and matrices $A_1, \ldots, A_k \in \mathcal{M}$ with $\rho(A_k \ldots A_1) \geq \rho(\mathcal{M})(1 - \frac{\Lambda_d}{n})$. By the definition of the Hausdorff distance there are matrices $\tilde{A}_1, \ldots, \tilde{A}_k \in \mathcal{N}$ with $\left\| A_i - \tilde{A}_i \right\| \leq \varepsilon, i = 1, \ldots, k$. To simplify notation, set $S_k := A_k \ldots A_1$ and $\tilde{S}_k := \tilde{A}_k \ldots \tilde{A}_1$.

Using the lemmas collected so far, we get the following inequalities where $c_1, \ldots, c_5$ are appropriately chosen constants only depending on $\mathcal{M}$, d and not on $\mathcal{N}$, n_0 or ε.

By the Berger-Wang theorem, (Berger and Wang, 1992, Theorem IV), we have

$$\rho(\mathcal{N})^k \geq \rho(\tilde{S}_k).$$

By Elsner's theorem we get

$$\rho(\mathcal{N})^k \geq \rho(S_k) - (\|\tilde{S}_k\|_2 + \|S_k\|_2)^{(d-1)/d} \|\tilde{S}_k - S_k\|_2^{1/d}.$$

For n_0 sufficiently large we can apply Lemma 9 and continue with

$$\begin{aligned} \rho(\mathcal{N})^k &\geq (1 - \frac{c_1}{n})^k - c_2 (k^{d-1})^{(d-1)/d} (\varepsilon k^{2d-1})^{1/d} \\ &= (1 - \frac{c_1}{n})^k - c_2 (\varepsilon k^{d^2})^{1/d} \\ &\geq (1 - \frac{c_1}{n})^k - c_2 (\varepsilon n^{d^2})^{1/d} \\ &\geq (1 - \frac{c_1}{n})^k - \frac{c_3}{n}. \end{aligned}$$

Depending on c_1 and c_3 we can now choose n_0 large enough such that a short calculation gives us

$$\rho(\mathcal{N}) \geq 1 - \frac{c_4}{n} \geq \rho(\mathcal{M}) - c_5 \varepsilon^{1/(d^2+d)}.$$

Given the choice of ε in (13) this yields the assertion. □

Combining Theorem 10 with Theorem 6 we get:

Corollary 11. Let $\mathcal{M} \in \mathcal{H}(d)$. For every $\eta > 0$ there is a constant C_η such that

$$|\rho(\mathcal{N}) - \rho(\mathcal{M})| \leq C_\eta d_H(\mathcal{M}, \mathcal{N})^{1/(d^2+d)}$$

for all $\mathcal{N} \in \mathcal{H}(d)$ with $d_H(\mathcal{M}, \mathcal{N}) \leq \eta$.

This is just another formulation of Theorem 3.

For finite sets of matrices $\mathcal{M}$, Morris obtained in Morris (2010) a better convergence speed in Proposition 7 of the form $O(n^\alpha)$ for arbitrary real $\alpha > 0$. With this estimate it is possible to improve the Hölder exponent for finite sets of matrices to $\frac{1}{d+\varepsilon}$ for arbitrary $\varepsilon > 0$, see Epperlein and Wirth (2023) for details.

5. CONCLUSIONS

In this paper we have continued the study of the regularity of the joint spectral radius as a function of the data, i.e. the defining matrix set. It has been shown that the joint spectral radius is pointwise Hölder continuous with a constant $1/(d^2 + d)$. We expect that better results can be obtained. On the one hand it is just the lower bound of our estimates which is responsible for the degradation of the constant.

On the other hand it would be much more desirable to obtain results on local Hölder continuity. In the context of ε-inflation this can be done, but this is a very restricted setting. In dimension 2, it is possible to obtain local Hölder continuity by a more careful tracking of the dependence of the constants and using an improved version of Lemma 8. We refer to Epperlein and Wirth (2023) for this statement. The extension of this result to higher dimensions is the topic of ongoing research.

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